DIHEDRAL SYMMETRIES OF MULTIPLE LOGARITHMS

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Abstract. This paper finds relationships between multiple logarithms with a dihedral group action on the arguments. I generalize the combinatorics developed in Gangl, Goncharov and Levin's R-deco polygon representation of multiple logarithms to find these relations. By writing multiple logarithms as iterated integrals, my arguments are valid for iterated integrals as over an arbitrary field.

Contents

This paper develops a relationship between different multiple logarithms that differ by a dihedral permutation on the arguments. The multiple logarithm

$$\mathbb{L}\mathrm{i}_{\underbrace{1,\ldots,1}_{r \text{ times}}}(x_1,\ldots x_r) = \sum_{0<k_1<\ldots<k_r} \frac{x_1^{k_1}\ldots x_r^{k_r}}{k_1\ldots k_r}$$

is a multiple polylogarithm $\mathbb{L}\mathrm{i}_{n_1\ldots n_r}(x_1,\ldots x_r)$ evaluated at $n_i = 1$. The multiple polylogarithm, evaluated at $x_i = 1$ gives the multiple zeta value $\zeta(n_1,\ldots n_r)$. Let D_{2r} be the dihedral group on r elements,

$$D_{2r} = \langle \sigma, \tau | \tau^2 = \sigma^r = 1, \sigma\tau = \tau\sigma^{-1}\rangle \ .$$

In this paper, I study the relationship between the multiple logarithms

$$\mathbb{L}\mathrm{i}_{\underbrace{1,\ldots,1}_{r \text{ times}}}(x_1,\ldots,x_r) \quad \text{and} \quad \mathbb{L}\mathrm{i}_{\underbrace{1,\ldots,1}_{r \text{ times}}}(g(x_1,\ldots,x_r))$$

for any $g \in D_{2r}$.

There is a Hopf algebraic structure to multiple polylogarithms, defined by studying them as iterated integrals over $\mathbb{C}$ with r marked points [6]. The multiple logarithms form a sub Hopf algebra. In [5] the authors establish a representation of multiple logarithms as decorated rooted oriented polygons, R-deco polygons. They establish a coalgebra homomorphism between a Hopf algebra built on these polygons and the Hopf algebra of multiple logarithms. The polygon associated to $\mathbb{L}\mathrm{i}_{\underbrace{1,\ldots,1}_{r \text{ times}}}(\frac{x_2}{x_1},\ldots \frac{x_{r+1}}{x_r})$ has edges labeled x_1 to x_{r+1}, with the root side x_{r+1}. The R-deco polygons generate a vector space that is graded by the

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weight of the corresponding multiple logarithm. Each polygon is endowed with a set of dissections. The authors define an algorithm that assigns to these dissections a dual tree and a differential, ∂, on the exterior algebra of these polygons $\mathcal{P}(R)$ to these dissections. The trees form a Hopf algebra, and the differential defines the bar construction $B_\partial(\mathcal{P})$ of iterated integrals construction introduced by Chen [3]. The polygon symmetries underlying this map makes it an ideal case to study dihedral group action on the arguments of multiple logarithms.

The algorithm above that assigns trees and differentials to each polygon is one of many possible choices of such algorithms. In this paper, I am interested in algorithms that either define a Hopf algebra of trees, or a specific perturbation off that algorithm. Each algorithm is chosen to exploit different symmetries of an action by a generator σ or τ of D_{2r}. These algorithms define different differential operators, which define different bar constructions on $\mathcal{P}(R)$. The combinatorics of the defined trees are more easily related that the combinatorics of the iterated integrals. Comparing tree structures gives relationships between multiple logarithms with dihedral group action on the arguments.

The dihedral group on three elements is the same as the symmetric group on three elements. Since the combinatorial arguments in this paper are inductive in nature, I hope to use this fact to relate multiple logarithms that differ by any permutation of the arguments.

There is a motivic generalization of Chen's iterated integrals. Bloch and Kritz [2] define a Hopf algebra of algebraic cycles, χ_{Mot}, over a field F formed by taking the 0^{th} cohomology of a bar complex based on a DGA associated to the cycles. In this context, the iterated integrals $I(0, x_1, \ldots, x_n, x_{n+1})$ are calculated over the algebraic cycle Δ_γ, and $x_i \in F$. In [5], the authors determine that there are elements of χ_{Mot} that correspond to multiple polylogarithms. The underlying combinatorics of the motivic multiple polylogarithms and the classical ones is the same. The results in this paper for comparing values of multiple logarithms with permuted arguments holds in both cases.

Section 1 of this paper generalizes the results of loc. cit. and work with multiple DGAs associated to the polygons. First I generalize the type of tree I allow, to allow for multi-rooted trees. I define a coassociative coproduct structure on the algebra of these generalized trees $\mathfrak{T}^\bullet(R)$. Then I generalize the algorithm defining dual trees to one that assigns a (now multi-rooted) tree to a dissection for a polygon, ϕ. I show that if the algorithm defines a sub-Hopf algebra of $\mathfrak{T}^\bullet(R)$, these trees define a differential ∂_ϕ on $\mathcal{P}(R)$. There is a bi-algebra homomorphism from the bi-algebra of trees to the bi-algebra underlying the bar construction $B_{\partial_\phi}(\mathcal{P}(R))$. In particular, I study the two different differentials on the R-deco polygons, ∂ and $\bar{\partial}$, defined in loc. cit. I define a class of perturbations of ϕ, ϕ', so that the associated differentials are the same $\partial_\phi = \partial_{\phi'}$, but where ϕ' need not defined a sub-Hopf algebra of trees. Finally, however, I show that the corresponding terms in $B_{\partial_\phi}(\mathcal{P}(R))$ do form a sub-Hopf algebra.

It is the difference between the algorithms or the form ϕ and ϕ' that gives the action of the dihedral group on R-deco polygons. Let $\Lambda(P)$ indicate the element in $B_\partial(\mathcal{P}(R))$, Chen's bar construction, corresponding to the R-deco polygon P. In section 2 I calculate the effect of the rotation and reflection maps, τ and σ on P. That is, I calculate $\Lambda(P) \pm \Lambda(\tau P)$ and $\Lambda(P) - \Lambda(\sigma P)$. By comparing with known properties of multiple logarithms, I show that the coalgebra map from Λ to $I(R)$ defined by [5] is not injective.

1. Bar constructions for R-deco polygons

This paper studies multiple logarithms by studying the iterated integral associated to them. Consider the 1-forms $\frac{dt_1}{t_1 - x_1}, \ldots \frac{dt_n}{t_n - x_n}$ on $M = \mathbb{C} \setminus \{x_1, \ldots x_{n+1}\}$, with $x_i \neq 0$ and $x_1 \neq x_j$. For a path γ from 0 to $x_{n+1} \in M$, the associated iterated integral is

$$I(0, x_1, \ldots, x_{n+1}) =$$
$$\int_{\Delta\gamma} \frac{dt_1}{t_1 - x_1} \wedge \ldots \wedge \frac{dt_n}{t_n - x_n} = \int_{0=t_0<t_1<\ldots<t_n<t_{n+1}=1} \frac{\gamma'(t_1)dt_1}{\gamma(t_1) - x_1} \cdots \frac{\gamma'(t_n)dt_n}{\gamma(t_n) - x_n} \, .$$

The value of this integral depends on the homotopy class of γ [3]. If γ is a path such that $\gamma(t)$ is a straight line from x_i to x_{i+1} for $t \in [t_i, t_{i+1}]$ then these iterated integrals can be related to multiple logarithms

$$(-1)^n I(0, x_1, \ldots x_n, x_{n+1}) = \mathbb{L}\mathrm{i}_{\underbrace{1, \ldots, 1}_{n \text{ times}}}(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_{n-1}}, \frac{x_{n+1}}{x_n}) \, .$$

These iterated integrals live in the 0^{th} cohomology of the associated bar complex defined by Chen. In this paper, I denote this bar complex $B_\partial(\mathcal{P}(R))$. The larger class of iterated integrals, $I(R)$ have a Hopf algebra structure, [6]. The author further showed that these iterated integrals have a motivic counterpart, $I^{\mathcal{M}}(0, x_1, \ldots, x_{n+1})$ with $x_i \in F$ for a field x that is an element of the fundamental motivic Tate Hopf algebra over F.

In [5], the authors defined a Hopf algebra on R-deco polygons, called $\Lambda_{\phi_2} \subset B_\partial(\mathcal{P}(R))$ in this paper. They then showed that there is an coalgebra homomorphism from this to $I(R)$,

$$\Phi : \Lambda_{\phi_2} \to I(R) \; .$$

If the polygon P has sides labeled $\{x_1, \ldots, x_{n+1}\}$ then

$$\Phi(\Lambda_{\phi_2}(P)) = I(0, x_1, \ldots, x_{n+1}) \; .$$

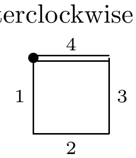
Relating dihedral symmetries of multiple logarithms can be simplified to a combinatorial problem on the dihedral symmetries of decorated polygons.

This section defines a class of Hopf algebras associated to these decorated polygons that are useful in solving the combinatorics of the how the polylogarithms vary as the order of the arguments are changed. Subsection 1.1 gives a definition of R-deco polygons, the vector space they generate, $V(R)$, its exterior product algebra $\mathcal{P}(R)$, and the associated bar constructions. Subsection 1.2 defines the algebra of multi-rooted trees, and the linearization map Λ. Here I show that Λ is a bialgebra homomorphism to the algebra of words on R-deco polygons. Subsection 1.3 defines dissections on polygons, and a class of algorithms that assign trees to these dissections. Subsection 1.4, shows that the algorithms that define Hopf algebras of trees, and certain variations on these algorithms define differentials on $\mathcal{P}(R)$. Finally, subsection 1.5 defines a relationship between the linearizations of algorithms that define the same differential. Let ϕ and ϕ' be two algorithms that both define the same differential ∂_ϕ, and let ϕ define a Hopf algebra of trees. Let $\Lambda_\phi, \Lambda_{\phi'} \subset B_{\partial_\phi}(\mathcal{P}(R))$ be the linearizations of the algebras of trees they define. I show that Λ_ϕ is a Hopf algebra, even if the algebra of trees it linearizes is not.

1.1. **Bar constructions on R-deco.** Let R be a set.

Definition 1. Let P_n be the convex oriented polygon with $n+1 \geq 2$ sides, with sides labeled by elements in R. One of those sides is a distinguished side, called a root side. One of the endpoints of the root side is marked as the first vertex. Orient P_n by starting at the first vertex and ending at the root side. The polygon P_n is an R-deco polygon, as defined in [5].

In this paper, I draw our polygons to be oriented counterclockwise. I sometimes specify a polygon, in terms of its labels, proceeding counterclockwise and ending with the root side. Therefore,



$= 1234$

The R-deco polygons generate a vector space.

Definition 2. Let $V_\bullet(R)$ be the graded vector space over $\mathbb{Q}$ generated by R-deco polygons. Let $V_n(R)$ be the vector space over $\mathbb{Q}$ generated by R deco $n+1$-gons, with $n > 0$ and $V_0(R)$ identified with $\mathbb{Q}$

$$V_\bullet(R) = \mathbb{Q} < \{1, P | P \text{ is a } R\text{-deco polygon}\} >= \oplus_{n=0}^{\infty} V_n(R) \quad ; \quad V_0(R) = \mathbb{Q}.$$

The *weight* of an element in $V_n(R)$ is n.

Definition 3. Let $\mathcal{P}_\bullet^{(\star)}(R)$ be the exterior product algebra of $V(R)$. It is bigraded, with a polynomial grading (weight) denoted by the subscript $\bullet$, and an exterior product grading (degree) denoted by the superscript $(\star)$.

The algebra $\mathcal{P}_\bullet^{(\star)}(R)$ can be endowed with a degree 1 differential operator to form a DGA $(\mathcal{P}_\bullet^{(\star)}(R), \partial)$. There are several such operators on this algebra, which I discuss in section 1.4. I consider the bar constructions associated to each DGAs, $B_\partial(\mathcal{P})$.

Definition 4. Let $(\mathcal{A}, \partial)$ be a DGA with $\mathcal{A}$ a graded exterior product algebra, and ∂ a degree 1 differential operator. The bar construction $B_\partial(\mathcal{A})$ associated to $(\mathcal{A}_\bullet, \partial)$ is the vector space underlying the tensor algebra $T(\mathcal{A}_\bullet) = \oplus_{i=1}^{\infty} \mathcal{A}_\bullet^{|i}$, with tensor symbol denoted by $|$ endowed with a commutative product, shuffle product $\sqcup\!\sqcup$. The bi-complex structure of $B_\partial(\mathcal{A})$ is given by the differential operators D_1 and D_2.

The coproduct on $B_\partial(\mathcal{A})$ is defined

$$\Delta[a_1|\dots|a_n] = \sum_{i=0}^{n} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_n] \,. \tag{1}$$

It is compatible with the shuffle product on $B_\partial(\mathcal{A})$.

In this paper, I consider $\mathcal{A}_\bullet = \mathcal{P}_\bullet^{(\star)}(R)$. Given a differential operator ∂, the bar construction $B_\partial(\mathcal{P}_\bullet^{(\star)}(R))$ is generated by terms of the form $[a_1|\dots|a_n]$ where each $a_i \in \mathcal{P}_\bullet^{(k_i)}(R)$ is homogeneous in the exterior product grading of degree k_i.

(1) Let $D_1 : \mathcal{P}(R)^{|n} \to \mathcal{P}(R)^{|n-1}$ be the operator defined

$$D_1([a_1|\dots|a_n]) = \sum_{i=1}^{n-1} (-1)^{\sum_{j\le i}(\deg\, a_j - 1)} [a_1|\dots|a_i \wedge a_{i+1}|\dots|a_n] \,.$$

(2) Let $D_2 : \mathcal{P}(R)^{|n} \to \mathcal{P}(R)^{|n}$ be the operator defined

$$D_2([a_1|\dots|a_n]) = \sum_{j=1}^{n} (-1)^{\sum_{k<j}(\deg\, a_k - 1)} [a_1|\dots|\partial a_j|\dots a_n] \,.$$

Since D_1 does not involve the differential defining the DGA, this differential is the same for all $B_\partial(\mathcal{P})$. If ∂ and ∂' are different differential operators on $\mathcal{P}_\bullet^{(*)}$, the differential D_2 is different on $B_\partial(\mathcal{P}(R))$ and $B_{\partial'}(\mathcal{P}(R))$.

1.2. **Trees.**

Definition 5. A tree is a finite contractible graph with oriented edges. Vertexes with all edges flowing away from it is called a root. Vertexes with all edges flowing into it are called leaves. A tree may have many roots, in which case is called a multi-rooted tree. If a tree has a single vertex, that vertex is both a root and a leaf.

Let $\mathfrak{T}^\bullet(R)$ be the augmented bialgebra over $\mathbb{Q}$ of multi-rooted trees with vertexes decorated by R-deco polygons. As with trees, a multi-rooted tree $T \in \mathfrak{T}^\bullet(R)$ induces a *partial order* on its vertices's. If v_1 and v_2 are two vertices's of a tree T, $v_1 \prec v_2$ in $T \iff \exists$ a path in T (defined by the orientation of the edges) flowing from v_1 to v_2. A *linear order* of T is a total ordering of the vertices's of T that respects the partial order. Unlike for single rooted trees, leaves on multi-rooted trees can have multiple edges coming into it.

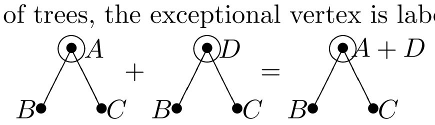
The algebra structure of $\mathfrak{T}^\bullet(R)$ is given as follows. It is graded by number of vertices's in the tree

$$\mathfrak{T}^\bullet(R) = \bigoplus_{n=0^\infty} \mathfrak{T}^n(R) = \mathbb{Q}\langle T | T \text{ has } n \text{ vertexes } \rangle \quad ; \quad \mathfrak{T}^0(R) = \mathbb{Q} \,.$$

The unit is the empty tree,

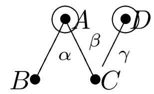
$$\mathbb{I}_{\mathfrak{T}}^\bullet(R) = T_\emptyset = \mathbb{I}_{\mathbb{Q}} \,.$$

The sum of two trees T_1 and T_2 is formal, unless they have the same underlying structure (the trees without vertex labels are isomorphic) and all but one corresponding vertices's have the same labels. In this case, in the sum of trees, the exceptional vertex is labeled by the sum of the vertices's.



The algebra $\mathfrak{T}^\bullet(R)$ is a commutative algebra with the product of trees being the disjoint union of trees, or a forest.

Definition 6. For a tree $T \in \mathfrak{T}^\bullet(R)$, let c be a subset of edges of T, and $\{t_1, \dots t_k\}$ be the set of trees formed by removing the edges in c. The subset c is an *admissible cut* of T if, for any individual t_i, the edges of c that have endpoints t_i either all flow into t_i or all flow from t_i.

Example 1. For example for the tree $T =$  the set $\{\alpha, \gamma\}$ is not an admissible cut, but the set $\{\alpha, \beta\}$ is.

Let c be an admissible cut of $\mathfrak{T}^\bullet(R)$, and $\{t_1, \ldots, t_k\}$ the set of subtrees of T formed by removing the edges in c from T. This set can be partitioned into two sets $\{t_{l_1}, \ldots, t_{l_n}\}$, the subtrees such that elements of c have at most a final point in t_{l_i}, and $\{t_{r_1}, \ldots, t_{r_m}\}$, the subtrees such that elements of c have at most a starting point in t_{r_j}.

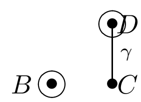
Definition 7. The pruned forest of an admissible cut is

$$L(c) = \prod_{i=1}^{n} t_{l_i}$$

and the root forest is

$$R(c) = \prod_{i=1}^{m} t_{r_i} \ .$$

In the above example, for $c = \{\alpha, \beta\}$, the pruned forest is

$L(c) =$ 

and the root forest is

$R(c) =$ 

Definition 8. The coproduct on $\mathfrak{T}^\bullet(R)$ is defined

$$\Delta T = \sum_{c \text{ admis.}} R(c) \otimes L(c) \ . \tag{2}$$

I denote the contribution of the admissible cut c to the coproduct as

$$\Delta_c(T) = R(c) \otimes L(c) \ .$$

In this notation $\Delta(T) = \sum_{c \text{ admis.}} \Delta_c(T)$.

Lemma 1. *The algebra $\mathfrak{T}^\bullet(R)$ is a coassociative Hopf algebra.*

Proof. Since $\mathfrak{T}^\bullet(R)$ is connected and graded, if it is a bialgebra, it is a Hopf algebra.

First I show that $\mathfrak{T}^\bullet(R)$ is a bialgebra. The coproduct defined in (2) is compatible with multiplication on $\mathfrak{T}^\bullet(R)$:

$$\Delta(TS) = \Delta(T)\Delta(S)$$

for $S, T \in \mathfrak{T}^\bullet(R)$. Let L_T, L_S be the pruned forests of T and S, and R_T and R_S the root forests of T and S. Then

$$\Delta(T)\Delta(S) = \sum_{d \text{ admis. of } T} \sum_{c \text{ admis. of } S} R_S(c) R_T(d) \otimes L_S(c) L_T(d) \ .$$

Since the product of trees is the disjoint union, an admissible cut of TS is an element of the form (d, c), where d admissible cut of T, and c is an admissible cut of S. Therefore,

$$\Delta(TS) = \sum_{(d,c) \text{ admis. of } TS} R_S(c) R_T(d) \otimes L_S(c) L_T(d) = \Delta(T)\Delta(S) \ .$$

Coassociativity means that for every $T \in \mathfrak{T}^\bullet(R)$,

$$(\Delta \otimes \mathbb{I})\Delta(T) = (\mathbb{I} \otimes \Delta)\Delta(T). \tag{3}$$

Let c be an admissible cut of T. Write

$$\Delta_c(T) = R(c) \otimes L(c) \ .$$

Let c_r be an admissible cut of the forest $R(c)$. Since the trees in the forest $R(c)$ are subtrees of T, c_r is also an admissible cut of T,

$$\Delta_{c_r}(T) = R(c_r) \otimes L(c_r) \ ,$$

and c is an admissible cut of $L(c_r)$. Similarly, if c_l is an admissible cut of $L(c)$, it is an admissible cut of T with

$$\Delta_{c_l}(T) = R(c_l) \otimes L(c_l) \ ,$$

and c is an admissible cut of $R(c_l)$. Therefore, for every admissible cut c, there exists another admissible cut of T, c_l (c_r) such that c is an admissible cut of the root (pruned) forest of c.

Explicitly write (3) as

$$(\Delta \otimes \mathbb{I})\Delta(T) = \sum_{c \text{ admis. of } T} \sum_{c_r \text{ admis. of } R(c)} \Delta_{c_r}(R(c)) \otimes L(c)$$
$$= \sum_{c_r \text{ admis. of } T} R(c_r) \otimes \sum_{c \text{ admis. of } L(c_r)} \Delta_c(L(c_r)) = (\mathbb{I} \otimes \Delta)\Delta(T).$$

□

Remark 1. Notice that if T is a single rooted (planar) tree, the coproduct defined above matches the coproduct and definition of admissible cut in [4]. For a single rooted tree, $R(c)$ is always a tree. If T is multi-rooted, $R(c)$ may be a forest.

Definition 9. Let $W(R)$ be the associative bi-algebra formed on the vectorspace underlying the tensor algebra $T(V(R))$ with a commutative product given by the shuffle product.

The coproduct on $W(R)$ is the same as the coproduct on $B_\partial(\mathcal{P}^(*)_\bullet(R))$, as given in (1)

$$\Delta w_1 \otimes \ldots \otimes w_n = \sum_{i=0}^{n} (w_1 \otimes \ldots \otimes w_i) \otimes (w_{i+1} \otimes \ldots \otimes w_n).$$

There is a natural inclusion from $W(R)$ to the algebra underlying $B_\partial(\mathcal{P}^(*)_\bullet(R))$ without the differential structure. To see this, write

$$W(R) = T(V(R), Ш, Delta) = (\mathcal{P}^{(1)}_\bullet(R), Ш, \Delta) \ .$$

There is an algebra homomorphism from the algebra of trees, $\mathfrak{T}^\bullet(R)$ to the algebra of words $W(R)$ which identifies a linear order on T with a word in $W(R)$. I first need to define linearizations of trees.

Definition 10. For $T \in \mathcal{T}^n(R)$, a partial order preserving a linearization of T, is

$$\lambda = \lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_n \in W(R)$$

where each λ_i is an R deco polygon labeling a vertex of T. If $\lambda_i \prec \lambda_j$ as vertices in T, then $i < j$.

Let $\mathrm{Lin}(T)$ be the set of partial order preserving linearizations of trees. A forest in $\mathfrak{T}^\bullet(R)$ also represents a linear order on its vertices. The linearization of trees extends naturally to forests.

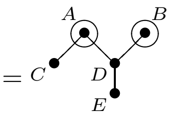
For any $\lambda \in \mathrm{Lin}(T)$, the polygon λ_1 is always the label of a root of T and λ_n is always the label of a leaf of T. For R-deco polygons P and Q, decorating the tree T, and let $\lambda_{i(P)}$ and $\lambda_{i(Q)}$ indicate the location of the polygons in the linearization λ of T. The vertexes $P \prec Q$ in T if and only if $i(P) < i(Q)$ for all $\lambda \in \mathrm{Lin}(T)$.

In this paper, the partial order of T is viewed as the sum of its linearizations. I define a map from trees to word by writing each tree as the sum of its linearizations:

$$\Lambda : \mathfrak{T}^\bullet(R) \to W(R)$$
$$T \mapsto \sum_{\mathrm{Lin}(T)} \lambda$$
$$T' \cdot T \mapsto \sum_{\mathrm{Lin}(T')} \lambda' \,Ш \sum_{\mathrm{Lin}(T)} \lambda = \Lambda(T) \,Ш\, \Lambda(T'),$$

where Ш is the shuffle product on $W(R)$.

Example 2. Let the tree T,

$T =$ 

has root vertexes A and B. Then

$$\lambda = A \otimes B \otimes C \otimes D \otimes E \quad \text{and} \quad \lambda' = \mathrm{B} \otimes \mathrm{A} \otimes \mathrm{D} \otimes \mathrm{E} \otimes \mathrm{C}$$

are two linear orders on T. The partial order represented by T is, in word form,

$$\Lambda(T) = (A \,\text{Ш}\, B) \otimes (C \,\text{Ш}\, (D \otimes E)) + A \otimes C \otimes B \otimes D \otimes E \ .$$

Theorem 1. *The map $\Lambda : \mathfrak{T}^\bullet(R) \to W(R)$ is a bi-algebra homomorphism.*

Proof. The algebra homomorphism comes from construction of the map Λ. The co-algebra homomorphism is harder to prove.

For $T \in \mathfrak{T}^n(R)$, the coproduct on T is

$$\Delta(T) = \sum_{c \text{ admis.}} R(c) \otimes L(c)$$

and the coproduct on the image, $\Lambda(T)$ is

$$\Delta\Lambda(T) = \sum_{i=0}^{n} \sum_{\lambda \in \mathrm{Lin}(T)} (\lambda_1 \otimes \ldots \otimes \lambda_i) \otimes (\lambda_{i+1} \otimes \ldots \otimes \lambda_n) \ .$$

Any decomposition of a linear order λ of T, $[\lambda_1 \otimes \ldots \otimes \lambda_i]$ and $[\lambda_{i+1} \otimes \ldots \otimes \lambda_n]$, can be written as a linear orders of forests of the form $\rho(R)$ and $\eta(L)$ with R and L sub-forests of T defined by the vertex sets $\{\lambda_1 \ldots \lambda_i\}$ and $\{\lambda_{i+1} \ldots \lambda_n\}$ respectively. The set of edges of T that connect the vertexes λ_j to λ_k for $j \leq i$ and $k > i$ define an admissible cut of T.

For each admissible cut c, the trees in the forests $L(c)$ and $R(c)$ are sub-trees of T. Let $\eta_c \in \mathrm{Lin}(L(C))$ and $\rho_c \in \mathrm{Lin}(R(c))$ be linear orders. Then

$$(\Lambda \otimes \Lambda) \circ \Delta(T) = \sum_{c \text{ admis.}} \sum_{\substack{(\eta_c, \rho_c) \in \\ \mathrm{Lin}(R(c)) \times \mathrm{Lin}(L(c))}} \rho_c \otimes \eta_c \ ,$$

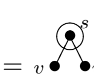
where the interior sum is taken over all linear orders of $R(c)$ and $L(c)$. By definition of admissible cut, each pair or linear orders $\rho_c \otimes \eta_c$, corresponds to a decomposition of a linear order λ of T, $[\lambda_1 \otimes \ldots \otimes \lambda_i] \otimes [\lambda_{i+1} \otimes \ldots \otimes \lambda_n]$ where the vertexes of $R(c)$ precede the vertexes of $L(c)$.

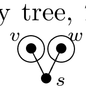
□

To complete the analysis in this paper, I need to introduce a method of grafting trees together by adding a new root or a new leaf. On the subalgebra of decorated single rooted trees in $\mathfrak{T}^\bullet(R)$, there is an operator on forests $B_+^s(\prod T_i)$ that defines a new tree by connecting the roots of the trees T_i to a new root with label s. This operator is discussed in [4] and [1], where it is shown to be a closed but not exact element of the first Hochschild cohomology on the complex defined by maps from $\mathcal{P}^{(1)} \to (\mathcal{P}^{(1)})^{\otimes n}$. This operator can be extended to $\mathfrak{T}^\bullet(R)$. In fact, given the symmetry between leaves and roots in multi rooted trees, one can also define an operator that joins trees by adding a new leaf instead of adding a new root.

Define two linear operators on the vectorspace underling $\mathfrak{T}^\bullet(R)$ that connect forests by adding new roots and leaves with label s respectively,

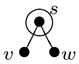
$$B_r^s, B_l^s : \mathbb{Q}\langle \text{forests with marked vertexes} \rangle \to \mathbb{Q}\langle \text{trees} \rangle \ .$$

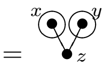
On the empty tree, $T_\emptyset = 1$, $B_r^s(T_\emptyset) = B_r^s(T_\emptyset) = \bullet^s$. On trees with one vertex, $B_r^s(\bullet^v, \bullet^w) =$ 

, and $B_l^s(\bullet^v, \bullet^w) =$ 

. For forests involving trees with multiple vertices's, there is not a natural choice of

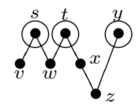
which vertex a root (or leaf) should be connected to in each tree. Therefore, it is necessary to specify a vertex for each tree in the forest. Let

$T_1 =$ 

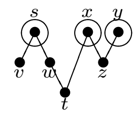
and $T_2 =$ 

.

Then

$B_r^t((T_1, w), (T_2, x)) =$ 

and

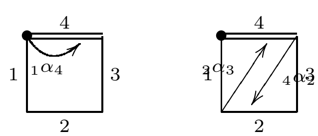
$B_l^t((T_1, w), (T_2, x)) =$ 

If, in the term $B_r^s(\prod(T_i, v_i))$, or $B_l^s(\prod(T_i, v_i))$, and $v_i \notin T_i$ for some i, then $B_r^s(\prod(T_i, v_i)) = 0$, or $B_l^s(\prod(T_i, v_i)) = 0$.

1.3. **From polygons to trees .** So far I have identified a Hopf algebra structure on the bar construction $B_\partial(\mathcal{P}_\bullet^{(*)}(R))$ for a DGA $(\mathcal{P}_\bullet^{(*)}(R), \partial)$, and on $\mathfrak{T}^\bullet(R)$. The coalgebra homomorphism Λ maps from $\mathfrak{T}^\bullet(R)$ to the tensor algebra underlying $B_\partial(\mathcal{P}_\bullet^{(*)}(R))$. I want to associate elements of $B_\partial(\mathcal{P}_\bullet^{(*)}(R))$, for a given ∂ with R-deco polygons. In this section I identify maps from $V(R)$ to sub-algebras of $\mathfrak{T}^\bullet(R)$. To each R-deco polygon generating $V(R)$, such a map associates a sum of trees in $\mathfrak{T}^\bullet(R)$. In section 1.4, I complete this association by associating differentials to subalgebras of $\mathfrak{T}^\bullet(R)$.

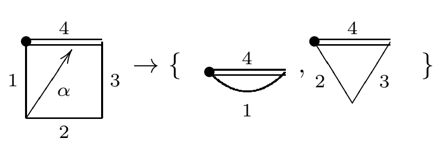
An R-deco polygon can be equipped with arrows, as in [5]. An arrow of a polygon is drawn from a vertex of a polygon to a side of a polygon. It divides the interior of the polygon into two regions. A trivial arrow of a polygon P goes from a vertex to an adjacent side. A non-trivial arrow of P is an arrow that does not end on a side adjacent to its starting vertex. Two arrows are said to be non-intersecting if they share not points in common other than possibly the starting vertex. Arrows in P_n are defined by their starting vertex and ending edge, ${}_i\alpha_j$, with $i, j \in 1 \dots n+1$. For non-trivial arrows, $j \neq i, i-1 \mod (n+1)$. Call ${}_i\alpha_j$ a backwards arrow if $j < i$. Otherwise it is a forwards arrow.

Example 3. The arrow ${}_1\alpha_4$ is a trivial arrow in the first polygon below. In the second polygon, ${}_2\alpha_4$ and ${}_4\alpha_2$ are non- intersecting, non-trivial arrows.



Regions associated to non-trivial dissection arrows can be views as polygons in their own right. If α, is a non-trivial arrow of P, contracting α to a point is a map from P to a set of two polygons $\{P_\alpha, Q_\alpha\}$ associated to the two regions of P. The labels of the sides and the orientations of P_α and Q_α are inherited from P. If α lands on a non-root side of P, then the subpolygon corresponding to the region that contains the root side of the original polygon inherits this root, and the side that α lands on becomes the new root for the other subpolygon. If α ends on the root side of P, then both subpolygons inherit the original root side as their root. Notice that $P_\alpha \in V_i(R)$ and $Q_\alpha \in V_{n-i}(R)$ for $P \in V_n(R)$.

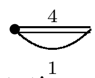
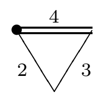
Example 4. For example, for $P = 1234$, and $\alpha = {}_2\alpha_4$ contracting along α gives the following map on set of polygons.



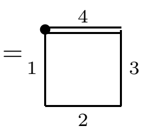
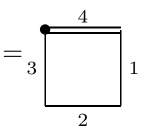
Definition 11. A dissection d of P is a non-intersecting set of non-trivial arrows of P. Denote $D(P)$ as the set of dissections of the polygon P, including the trivial dissection (no arrows). The cardinality of a dissection, $|d|$ is the number of non-trivial arrows in d.

The polygons P_α and Q_α above are called the polygons associated to the dissecting arrow α. If $d \in D(P)$ is a dissection with i arrows, there is a set of $i+1$ subpolygons, $\{P_0, \dots P_i\}$ associated to the dissection d, formed by contracting the arrows in d. If each $P_j \in V_{n_j}(R)$, and $P \in V_n(R)$, then $\sum_{j=0}^{i} n_j = n$.

For a dissection consisting of a single arrow, $\alpha = d \in D(P)$, the subpolygons associated to α are sometimes referred to as the *root polygon*, $P_\alpha^{\bullet}$, which is the subpolygon that contains the root side and first vertex of P, and the *cutoff polygon*, $P_\alpha^{\sqcup}$, which is the other subpolygon. At other times, it is convenient to consider whether the subpolygon lies to the left or the right of the arrow, as determined when the arrow is oriented upwards on the page. In this case, the *left polygon* is indicated P_α^l and the *right polygon* is indicated P_α^r. Notice that if α is a forwards arrow, $P_\alpha^l = P_\alpha^{\sqcup}$. If it is a backwards arrow, $P_\alpha^l = P_\alpha^{\bullet}$. In the previous example, since ${}_2\alpha_4$ is a forward arrow,

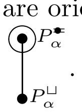
$P_\alpha^l = P_\alpha^{\bullet} =$  , $P_\alpha^r = P_\alpha^{\sqcup} =$ 

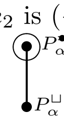
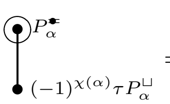
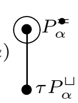
Let τ be a map that reverses the orientation of a polygon: if $P = r_1 \dots r_{n+1}$, with $r_i \in R$, $\tau(P) = r_{n-1} \dots r_1 r_n$.

$P =$  $\tau(P) =$ 

Definition 12. Define $\chi(\alpha)$ to be the weight of the cutoff polygon. That is, $P_\alpha^{\sqcup} \in \mathcal{P}^{(1)}_{\chi(\alpha)}$.

One can assign trees, $T_d(P) \in \mathfrak{T}^{\bullet}(R)$, to dissections of a polygon $d \in D(P)$ with $P \in V(R)$. The edges of $T_d(P)$ correspond to arrows in d. If $\alpha \in d$, such that P_1 and P_2 are the two polygons adjacent to α, one endpoint of the corresponding edge in $T_d(P)$ is labeled $\pm\tau^i P_1$ and the other $\pm\tau^j P_2$ for $j \in \{0, 1\}$. The polygons $\tau^i P_1$ are $\tau^j P_2$ are *adjacent* in $T_d(P)$. The specifics of how to assign a tree to a dissection is given by a map ϕ. I give four examples of such maps before beefing it formally.

Example 5. Let $T_{\phi_1,d}(P) \in \mathfrak{T}^{\bullet}(R)$ be the tree defined on the polygon P with the dissection $d \in D(P)$. Each vertex of the $T_{\phi_1,d}(P)$ is labeled with a subpolygon associated to a subpolygon associated to d. It has a single root vertex labeled by the subpolygon that contains the original root side and first vertex of P. The edges of the tree are oriented to flow away from the root vertex. Given a single dissecting arrow $d = \alpha \in D(P)$, $T_{\phi_1,\alpha}(P) =$  .

Example 6. Let $T_{\phi_2,d}(P) \in \mathfrak{T}^{\bullet}(R)$ be the tree defined on the polygon P with the dissection $d \in D(P)$. Each vertex of the $T_{\phi_2,d}(P)$ is labeled with a subpolygon, or the subpolygon with reverse orientation, associated to a subpolygon associated to d. It has a single root vertex corresponding to subpolygon that contains the original root side and first vertex of P. The edges of the tree are oriented to flow away from the root vertex. Let e_2 be the terminal vertex for the edge e in $T_{\phi_2,d}(P)$, and let Q be the polygon associated to d corresponding to e_2. If e corresponds to a backwards arrow in $d \in D(P)$, then the label of e_2 is $(-1)^{|Q|}\tau Q$. Otherwise, it is labeled by Q. Given a single dissecting arrow $d = \alpha \in D(P)$, $T_{\phi_2,\alpha}(P) =$  if α is a forward arrow, and $T_{\phi_2,\alpha}(P) =$  $= (-1)^{\chi(\alpha)}$  .

Example 7. Let $T_{\phi_3,d}(P) \in \mathfrak{T}^{\bullet}(R)$ be the tree defined on the polygon P with the dissection $d \in D(P)$. Each vertex of the $T_{\phi_3,d}(P)$ is labeled with a subpolygon, associated to a subpolygon associated to d. It has a single root vertex corresponding to subpolygon that contains the original root side and first vertex of P. The

edges of the tree are oriented to flow away from the root vertex. Let e_2 be the terminal vertex for the edge e in $T_{\phi_3,d}(P)$, and let Q be the polygon associated to d corresponding to e_2. If e corresponds to a backwards arrow in $d \in D(P)$, then the label of e_2 is $(-1)Q$. Otherwise, it is labeled by Q. Given a single dissecting arrow $d = \alpha \in D(P)$, $T_{\phi_3,\alpha}(P) = \begin{array}{l} \odot P_\alpha^{\bullet} \\ | \\ \bullet P_\alpha^{\sqcup} \end{array}$ if α is a forward arrow, and $T_{\phi_3,\alpha}(P) = \begin{array}{l} \odot P_\alpha^{\bullet} \\ | \\ \bullet -P_\alpha^{\sqcup} \end{array} = - \begin{array}{l} \odot P_\alpha^{\bullet} \\ | \\ \bullet P_\alpha^{\sqcup} \end{array}$.

Example 8. Let $T_{\phi_4,d}(P) \in \mathfrak{T}^\bullet(R)$ be the tree defined on the polygon P with the dissection $d \in D(P)$. Each vertex of the $T_{\phi_4,d}(P)$ is labeled with a subpolygon associated to a subpolygon associated to d. It is a multi rooted tree, with edges oriented to flow from the region to the left of the arrow to the right of the arrow. Left and right are determined from the point of view of arrow being oriented up on the page. Given a single dissecting arrow $d = \alpha \in D(P)$, $T_{\phi_4,\alpha}(P) = \begin{array}{l} \odot P_\alpha^{l} \\ | \\ \bullet P_\alpha^{r} \end{array}$.

There are many ways to assign a tree to a dissection of a polygon such that the edges correspond to the arrows in the dissection, and the vertexes to the polygons associated to the dissection. The examples above are only a few examples of

Definition 13. A rule assigning a tree to a dissection of a polygon is a map

$$\begin{aligned} \phi : \{d \in D(P) | P \in V(R)\} &\to \mathfrak{T}^\bullet(R) \\ d \in D(P) &\to T_{\phi,d}(P) \end{aligned}$$

such that the edges of $\phi(d, P)$ correspond to the arrows in the dissection d. If $\alpha \in d$ is an arrow that separates the regions P_1 and P_2 in P, then the corresponding edge in $T_{\phi,d}(P)$ connects the vertexes v_1 and v_2, where

$$v_i \in \{(-1)^k \tau^j P_i | j, k \in \{0, 1\}\} \ .$$

The rule ϕ determines the value of j, k for each dissection d of every polygon P as well as whether the edge corresponding to α flows from v_1 to v_2 or from v_2 to v_1.

Explicitly, a rule maps from a dissection of a polygon to a tree, assigning a sign and orientation to each subpolygon associated to the dissection, and a partial order to the set of subpolygons.

I extend rules to define linear maps from the vector space generated by R-deco polygon to the vector space generated by trees

$$\begin{aligned} \phi : V(R) &\to \mathfrak{T}^\bullet(R) \\ P &\to \sum_{d \in D(P)} T_{\phi,d}(P) \ . \end{aligned}$$

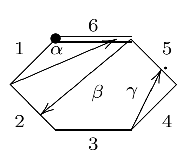
Let $\mathfrak{T}_\phi$ be the subalgebra of $\mathfrak{T}^\bullet(R)$ generated by the vector space $\phi(V(R))$. In this paper, I am interested in rules ϕ that generate Hopf algebras $\mathfrak{T}_\phi$, and a family of closely related maps.

Definition 14. Let $v = \{v_1, \ldots, v_k\}$ be the set of vertexes of the tree $T_{\phi,d}(P)$. The sign of a dissection, $d \in D(P)$, as determined by the rule ϕ is

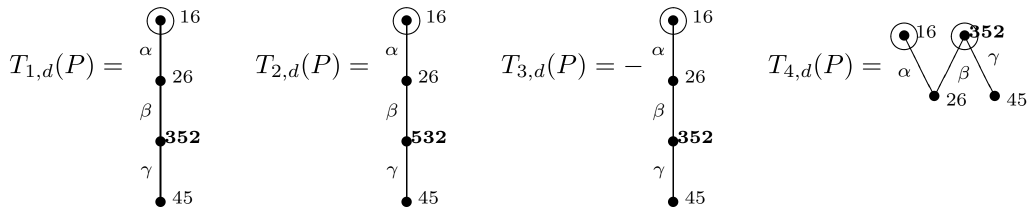
$$\text{sign}_\phi(d) = \prod_{v_i \in v} \text{sign}(v_i) = \text{sign}(T_d(P)) \ .$$

For $i \in \{1 \ldots 4\}$, let $\mathfrak{T}_i$ to be the sub-algebra of $\mathfrak{T}^\bullet(R)$ generated by the maps $\phi_i(V(R))$ in the above examples. The following example illustrates the differences in the trees defined by the rules ϕ_i.

Example 9. Let

$P =$ 

Then the associated trees are



The tree structures in examples 6 and 7 are defined in [5]. The authors prove that $\mathfrak{T}_2$ is a Hopf algebra. It remains to check that $\mathfrak{T}_1$, $\mathfrak{T}_3$ and $\mathfrak{T}_4$ are Hopf algebras.

Lemma 2. *The algebras $\mathfrak{T}_1$, $\mathfrak{T}_3$ and $\mathfrak{T}_4$ are Hopf algebras.*

Proof. It is sufficient to show that these are bi-algebras.

For T_4, consider the set of trees $\{T_{4,d}(P)|P \in V(R), d \in D(P)\}$. First consider an admissible cut of one arrow, $c = \alpha$. For each $\alpha \in d$, suppose the subpolygon P_i lies to the right (left) of $\beta \in d \setminus \alpha$. The set $d \setminus \alpha = d_l \cup d_r$ is a set of dissecting arrows in P^l_α and a set of dissecting arrows in P^r_α. Without loss of generality, assume β is a dissecting arrow of P^l_α. The subpolygon P_i continues to lie on the right (left) side of β. That is, structures of the sub-trees does not change after dissection: the root tree $R(\alpha) = T_{4,\alpha}(P^l_\alpha)$, and the pruned tree $L(\alpha) = T_{4,\alpha}(P^r_\alpha)$. For admissible cuts with multiple arrows, $c = \{\alpha_1, \ldots, \alpha_n\}$, first remove the edge dual to α_1. Then $R(\alpha_1) = T_{4,\alpha_1}(P^l_{\alpha_1})$, and the pruned tree $L(\alpha_1) = T_{4,\alpha}(P^r_{\alpha_1})$. The argument follows by induction, and is indifferent to the ordering of the arrows in c.

The argument is similar for $\mathfrak{T}_1$ and $\mathfrak{T}_3$. Consider the two sets of trees $\{T_{1,d}(P)|P \in V(R), d \in D(P)\}$ and $\{T_{3,d}(P)|P \in V(R), d \in D(P)\}$. First consider admissible cuts consisting of single arrows. For each $\alpha \in d$, suppose the subpolygon P_i lies on the root (cutoff) side of $\beta \in d \setminus \alpha$. The set $d \setminus \alpha = d_{\bullet\!-} \cup d_{\sqcup}$ is the union of a set of dissecting arrows in $P^{\bullet\!-}_\alpha$ and $P^{\sqcup}_\alpha$. Without loss of generality, assume β is a dissecting arrow of $P^{\bullet\!-}_\alpha$. The subpolygon P_i continues to lie on the root (cutoff) side of β. That is, structures of the sub-trees does not change after dissection. For $i \in \{1, 3\}$, the root tree $R(\alpha) = T_{i,\alpha}(P^{\bullet\!-}_\alpha)$, and the pruned tree $L(\alpha) = T_{i,\alpha}(P^{\sqcup}_\alpha)$. For admissible cuts with multiple arrows, $c = \{\alpha_1, \ldots, \alpha_n\}$, first remove the edge dual to α_1. Then $R(\alpha_1) = T_{i,\alpha_1}(P^{\bullet\!-}_{\alpha_1})$, and the pruned tree $L(\alpha_1) = T_{i,\alpha}(P^{\sqcup}_{\alpha_1})$. The argument follows by induction, and is indifferent to the ordering of the arrows in c. $\square$

Not all rules $\phi(V(R))$ generate sub Hopf algebras of $\mathfrak{T}^\bullet(R)$.

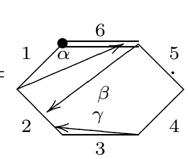
Definition 15. Let $re(P)$ be the set of arrows ending on the root side of an R-deco polygon P (the *r*oot *e*nding arrows). To fix notation, write $re(P) = \{{}_2\alpha, \ldots {}_{n-1}\alpha\}$, where ${}_i\alpha$ starts at the i^{th} vertex.

I define a rule ϕ_{re} such that

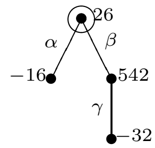
$$T_{\phi_{re},d}(P) = T_{\phi_2,d}(P) \quad \text{if} \quad d \cap re(P) = \emptyset \, .$$

If $\alpha \in d \cap re(P)$ with v_1 and v_2 the endpoints of the corresponding edge in $T_{\phi_2,d}(P)$ such that $v_1 \prec v_2$, then $-v_2 \prec v_1$ in $T_{\phi_{re},d}(P)$. That is the rule ϕ_{re} differs from ϕ_2 by the order and sign assigned to vertices adjacent to edges corresponding to arrows in $re(P)$. Let $\mathfrak{T}_{re}$ be the algebra generated by ϕ_{re}. The following is an example of a tree generated by this rule.

Example 10. For $d = \{\alpha, \beta, \gamma\}$ and

$P =$ 

The tree

$T_{re,d}(P) =$ 

For the polygon in example 10, consider the admissible cut corresponding to a non root ending edge $c = \alpha \notin re(P)$. If $\alpha \in d \in D(P)$, write $d = d_{\bullet\!-} \cup \alpha \cup d_{\sqcup}$, where $d_{\bullet\!-} \in D(P_\alpha^{\bullet\!-})$ and $d_{\sqcup} \in D(P_\alpha^{\sqcup})$. The corresponding term of the coproduct is

$$\Delta_c(P) = \sum_{\substack{d \in D(P) \\ d \supset \alpha}} T_{\phi_{re}, d_{\bullet\!-}}(P_\alpha^{\bullet\!-}) \otimes T_{\phi_2, d_{\sqcup}}(P_\alpha^{\sqcup}) = \phi_{re}(P_\alpha^{\bullet\!-}) \otimes \phi_2(P_\alpha^{\sqcup}) \ .$$

The second term in the tensor product is in $\mathfrak{T}_2$, not $\mathfrak{T}_{re}$. Therefore, $\mathfrak{T}^\bullet(R)_{re}$ is not a sub Hopf algebra of $\mathfrak{T}^\bullet(R)$.

1.4. **Trees to differentials.** So far, I have defined a set of linear maps

$$\phi : V(R) \to \mathfrak{T}^\bullet(R)$$

and a linearization map

$$\Lambda : \mathfrak{T}^\bullet(R) \to B_\partial(\mathcal{P}_\bullet^{(\star)}(R))$$

that assigns to each tree a element of the bar construction $B_\partial(\mathcal{P}_\bullet^{(\star)}(R))$, ignoring the differential structure.

Definition 16. Let $\Lambda_\phi = \Lambda \circ \phi$ be the composition that associates an element of the bar construction to each polygon.

I am interested in identifying pairs (ϕ, ∂) such that each polygon maps to an element of the zeroth cohomology of $B_\partial(\mathcal{P}_\bullet^{(\star)}(R))$

$$\Lambda_\phi(P) \in H^0(B_\partial(\mathcal{P}_\bullet^{(\star)}(R)) \ . \tag{4}$$

In this section, I show that if $\mathfrak{T}_\phi$ is a Hopf algebra generated by $\phi(V(R))$, then the rule ϕ also defines a differential operator, ∂_ϕ, on $\mathcal{P}_\bullet^{(*)}(R)$. Then the pair (ϕ, ∂_ϕ) satisfies condition (4).

To define ∂_ϕ, I need to additional maps. Let π_n be the projection of a vectorspace onto it's n^{th} graded component. Let q be the quotient map

$$q : T(V(R)) \to T(V(R))/(a \otimes b + b \otimes a) \ .$$

Then I can define an operator $\partial_\phi = q \circ \Lambda \circ \pi_2 \circ \phi$,

$$\partial_\phi : V(R) \to \mathcal{P}_\bullet^{(1)}(R) \ .$$

For any dissection $d \in D(P)$ such that $|d| = 1$, let P_d^1 and P_d^2 be the labels of the root and leaf vertexes of $T_{\phi,d}(P)$. Then

$$\partial_\phi(P) = \sum_{d \in D(P), |d|=1} P_d^1 \wedge P_d^2 \ .$$

I show that if ϕ defines a Hopf algebra, $\partial_\phi \circ \partial_\phi = 0$.

Theorem 2. *Let $\mathfrak{T}_\phi$ be a sub-Hopf algebra of $\mathfrak{T}^\bullet(R)$ generated by $\phi(V(R))$. For α a non-trivial dissecting arrow of P, let P_α^1 and P_α^2 be the labels of the initial and final vertexes of tree $T_{\phi,\alpha}(P)$. Then ϕ defines a degree one differential on $\mathcal{P}_\bullet^{(*)}(R)$,*

$$\partial_\phi(P) = \sum_{d \in D(P), |d|=1} P_d^1 \wedge P_d^2$$

under the Leibniz rule $\partial_\phi(a \wedge b) = (\partial_\phi a) \wedge b + (-1)^i a \wedge \partial_\phi(b)$, where $a \in \mathcal{P}_\bullet^{(i)}(R)$.

Proof. I show that that $\partial \circ \partial = 0$ using coassociativity of the coproduct on $\mathfrak{T}^{\bullet}(R)$. Coassociativity of the coproduct on multi-rooted trees can be written

$$\sum_c (\sum_{c_l} \Delta_{c_l} \otimes \mathbb{I}) \Delta_c = \sum_c (\sum_{c_r} \mathbb{I} \otimes \Delta_{c_2}) \Delta_c \ ,$$

where c, c_l and c_r are admissible cuts for P, $R(c)$ and $L(c)$ respectively. Specifically,

$$(\Delta_{c_l} \otimes \mathbb{I}) \Delta_c = (\mathbb{I} \otimes \Delta_{c_r}) \Delta_{c'}$$

if $c \cup c_l = c' \cup c_r$. For any dissection $d = \{\alpha, \beta\} \in D(P)$, write

$$\Delta_\alpha(L(\beta)) = L_L(\alpha) \otimes L_R(\alpha)$$

and

$$\Delta_\alpha(R(\beta)) = R_L(\alpha) \otimes R_R(\alpha) \ .$$

Then

$$L_L(\alpha) \otimes L_R(\alpha) \otimes R(\beta) = L(\beta) \otimes R_L(\alpha) \otimes R_L(\alpha) \ .$$

Composition of ∂_ϕ gives

$$\begin{aligned} \partial_\phi \circ \partial_\phi(P) &= \sum_\beta (\partial_\phi L(\beta)) \wedge R(\beta) - L(\beta) \wedge \partial_\phi(R(\beta)) \\ &= \sum_{\alpha, \beta} L_L(\alpha) \wedge L_R(\alpha) \wedge R(\beta) - L(\beta) \wedge R_L(\alpha) \wedge R_R(\alpha) \end{aligned}$$

which is 0 by the Leibniz rule and coassociativity. □

Since $\mathfrak{T}_1$, $\mathfrak{T}_2$, $\mathfrak{T}_3$, and $\mathfrak{T}_4$ are all Hopf algebras, they define degree 1 differential operators ∂_{ϕ_1}, ∂_{ϕ_2}, ∂_{ϕ_3} and ∂_{ϕ_4} on $\mathcal{P}_\bullet^{(\star)}$. These differentials are defined below.

Example 11. First I establish some notation for this example. Let

$$i(\alpha) = \begin{cases} 1 & \alpha \text{ backwards} \\ 0 & \alpha \text{ backwards} \end{cases} ,$$

and

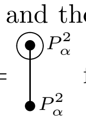
$$\text{sign}(\alpha) = \begin{cases} (-1)^{\chi(\alpha)} & \alpha \text{ backwards} \\ 1 & \alpha \text{ backwards} \end{cases} .$$

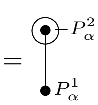
The following are the differentials defined by ϕ_i, the rules that generate the Hopf algebras $\mathfrak{T}_i$.

(1) $\partial_1(P) = \sum_{d \in D(P), |d|=1} P_d^{\bullet\!-} \wedge P_d^{\sqcup}$
(2) $\partial_2(P) = \sum_{d \in D(P), |d|=1} \text{sign}(d) P_d^{\bullet\!-} \wedge \tau^{i(d)} P_d^{\sqcup}$
(3) $\partial_3(P) = \sum_{d \in D(P), |d|=1} (-1)^{i(d)} P_d^{\bullet\!-} \wedge P_d^{\sqcup}$
(4) $\partial_4(P) = \sum_{d \in D(P), |d|=1} P_d^l \wedge P_d^r$

where the sums are taken over all non-trivial dissecting arrows α in P.

Notice that $\partial_3 = \partial_4$. If α is a backwards arrow, $P_\alpha^l = P_\alpha^{\sqcup}$. The four sub Hopf algebras of $\mathfrak{T}^{\bullet}(R)$ define 3 different differentials on $\mathcal{P}_\bullet^{(*)}(R)$. In this paper, I are primarily interested in the differentials ∂_2 and ∂_3.

In this paper, I are interested in different sub-algebras of $\mathfrak{T}^{\bullet}(R)$ that induce the same differential on $\mathcal{P}_\bullet^{(*)}(R)$. For this, I consider tree defining rules, ϕ' that differ from a set of rules ϕ that generates a sub-Hopf algebras of $\mathfrak{T}^{\bullet}(R)$ by the orientation of a fixed subset of edges and the sign associated to one of the endpoints of each of those edges in a tree. For instance, let $T_{\phi,\alpha}(P) =$ 

for all $\alpha \in s$, where s is a fixed subset of dissecting arrows of P. Then I are interested in rules ϕ' such that $T'_{\phi',\alpha}(P) =$ 

for all $\alpha \in s$. Furthermore, for all $\beta \notin s$, $\phi(\beta, P) = \phi'(\beta, P)$.

Lemma 3. *Let ∂_ϕ be the differential defined by ϕ. Let ϕ' be the rule that agrees with ϕ on all dissecting arrows outside the set s, and for all arrows $\alpha \in s$, $\phi(\alpha, P) = \begin{tikzcd}[row sep=small] \odot\, P_\alpha^1 \arrow[d, dash] \\ \bullet\, P_\alpha^2 \end{tikzcd}$ and $\phi'(\alpha, P) = \begin{tikzcd}[row sep=small] \odot\, P_\alpha^2 \arrow[d, dash] \\ \bullet\, P_\alpha^1 \end{tikzcd}$. Then ϕ and ϕ' define the same differential.*

Proof. Write

$$\partial = \sum_{d \in D(P), |d|=1, d \cap s = \emptyset} P_d^1 \wedge P_d^2 + \sum_{d \in D(P), |d|=1, d \in s} P_d^1 \wedge P_d^2 \ .$$

Meanwhile, the rule ϕ' induces a degree one operator

$$\sum_{d \in D(P), |d|=1, d \cap s = \emptyset} P_d^1 \wedge P_d^2 + \sum_{d \in D(P), |d|=1, d \in s} -P_d^2 \wedge P_d^1 \ ,$$

which is the same as ∂. □

Notice that the trees in $\mathfrak{T}_3$ and in $\mathfrak{T}_4$ differ by sign of terminal vertex and orientation of edges dual to backwards arrows. As noted above, $\partial_3 = \partial_4$. Similarly, the trees in the algebra $\mathfrak{T}_{re}$ differ from the trees in $\mathfrak{T}_2$ by the orientation of the edges dual to arrow that end on the root side and the sign of the terminal vertex. The algebra $\mathfrak{T}_{re}$ also induces ∂_2.

Finally, I show that $\Lambda(\mathfrak{T}_\phi) \in H^0(B_{\partial_\phi}(\mathcal{P}_\bullet^{(*)}(R))$. Recall that by identifying $W(R) = (T(\mathcal{P}_\bullet^{(1)}(R)), Ш\,, \Delta)$, the linearization of trees gives a bialgebra homomorphism

$$\Lambda : \mathfrak{T}^\bullet(R) \to (T(\mathcal{P}_\bullet^{(1)}(R)), Ш\,, \Delta) \ .$$

By specifying a differential operator on $\mathcal{P}_\bullet^{(*)}(R)$, I can write this as a map

$$(\Lambda, \partial) : \mathfrak{T}^\bullet(R) \to B_\partial(\mathcal{P}_\bullet^{(*)}(R)) \ .$$

Definition 17. If ϕ is a rule that defines the differential ∂_ϕ, and $\mathfrak{T}_\phi$ the algebra generated by the rule, write the corresponding subalgebra generated by generated by $\Lambda_\phi(V(R))$ as $\Lambda(\mathfrak{T}_\phi) \subset B_{\partial_\phi}(\mathcal{P}_\bullet^{(*)}(R))$.

Theorem 3. *Let ∂_ϕ be the differential on $\mathcal{P}_\bullet^{(*)}(R)$ defined by the rule ϕ, and $\mathfrak{T}_\phi$ be the sub algebra $\mathfrak{T}^\bullet(R)$ generated by $\phi(V(R))$. Every element of Λ_ϕ is a 0 co-cycle of $D_1 + D_2$ in $B_{\partial_\phi}(\mathcal{P}_\bullet^{(*)}(R))$.*

Proof. Let π_k be the projection of Λ_ϕ onto its k^{th} direct sum component,

$$\pi_k : \Lambda_\phi \to \mathcal{P}_\bullet^{(\star)}(R)^{|k} \ .$$

The proof proceeds by comparing D_1 and D_2 on the algebra generators, $\Lambda_\phi(P)$.

Since $\Lambda(\mathfrak{T}_\phi) \subset T(\mathcal{P}_\bullet^{(1)}(R))$, reduce the definitions of D_1 and D_2 to

$$D_1([a_1|\ldots|a_n]) = \sum_{i=1}^{n-1} (-1)^{i-1} [a_1|\ldots|a_i \wedge a_{i+1}|\ldots|a_n]$$

$$D_2([a_1|\ldots|a_n]) = \sum_{j=1}^{n} (-1)^{j-2} [a_1|\ldots|\partial_\phi(a_j)|\ldots|a_n]$$

If P is a polygon of weight n, $\pi_n \circ \Lambda_\phi(P)$ is a sum of n-fold tensors of 2 gons. Therefore $D_2(\pi_n \circ \Lambda_\phi(P)) = 0$. By construction, the term $D_1(\pi_1 \circ \Lambda_\phi(P)) = 0$.

Any tree dual to a dissection with $k-1$ arrows has k vertexes. Comparing $D_1(\pi_k \circ \Lambda_\phi(P))$ to $D_2(\pi_{k-1} \circ \Lambda_\phi(P))$ for $k \in \{2, \ldots, n\}$ gives the terms

$$D_1(p_k \circ \Lambda_\phi(P)) = \sum_{d \in D(P), |d|=k-1} \sum_{\mathrm{Lin.}(T_d(P))} \sum_{i=1}^{k} \mathrm{sign}(d)(-1)^{i-1} [\lambda_1|\ldots|\lambda_i \wedge \lambda_{i+1}|\ldots|\lambda_k]$$

versus

$$D_2(\pi_{k-1} \circ \Lambda_\phi(P)) = \sum_{d' \in D(P), |d'| = k-2} \sum_{\text{Lin.}(T_{d'}(P))} \sum_{i=1}^{k-1} \text{sign}(d')(-1)^{i-2} [\lambda'_1 | \dots | \partial \lambda'_i | \dots | \lambda'_{k-1}] \; .$$

If λ_i and λ_{i+1} are not adjacent in $T_{\phi,d}(P)$, then there exists a different linear order ρ of the same tree that switches the order and leaves the other terms the same, so that $\lambda_i = \rho_{i+1}$, $\lambda_{i+1} = \rho_i$ and $\lambda_j = \rho_j$ if $j \notin \{i, i+1\}$. The corresponding terms in the sum for D_1 cancel. In the remaining terms for D_1, when $\lambda_i \wedge \lambda_{i+1}$ appears, there exists a dissection $d \in D(P)$ such that $|d| = k-1$ where λ_i and λ_{i+1} are adjacent, with connecting edge dual to $\alpha \in d$ in $T_{\phi,d}(P)$. Each $\alpha \in d$ defines a $d' = d \setminus \alpha$. The term $\text{sign}(\alpha)\lambda_i \wedge \lambda_{i+1}$ appears in D_2 as wedge product induced by $(O, v)(\alpha)$ in differential ∂. Therefore the term $\lambda_i \wedge \lambda_{i+1}$ appears in the i^{th} tensor component of ∂_1 with sign $-\text{sign(d)}(-1)^i$ and in ∂_2 with sign $-\text{sign}(d')\text{sign}(\alpha)(-1)^i$. Since $\text{sign}(d) = \text{sign}(d')\text{sign}(\alpha)$, these terms appear with opposite signs and cancel.

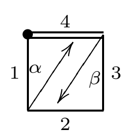
Next consider terms in Λ_ϕ that are the product of two generators, $\Lambda_\phi(P) \amalg \Lambda_\phi(Q)$, for $P \in V_n(R)$, and $Q \in V_m(R)$. The projection $\pi_1(\Lambda_\phi(P) \amalg \Lambda_\phi(Q)) = 0$, and $D_1(\pi_2(\Lambda_\phi(P)) \amalg \Lambda_\phi(Q))) = 0$. Furthermore, $D_2(\pi_{n+m}(\Lambda_\phi(P)) \amalg \Lambda_\phi(Q))) = 0$. As above, compare $D_1(\pi_k \circ \Lambda_\phi(P))$ to $D_2(\pi_{k-1} \circ \Lambda_\phi(P))$ for $k \in \{3, \dots, n\}$. The only new case to consider is if λ_i and λ_{i+1} are subpolygons of different trees, in which case there is another linear order that switches the order, canceling these terms.

Since $D_1 + D_2$ is a linear operator, this extends to the entire algebra. □

The coproduct on the sub Hopf algebras $\Lambda(\mathfrak{T}_\phi)$ have a particularly nice form. The duality between dissections $d \in D(P)$ and edges of a tree $T_{d,\phi}(P)$ for some rule ϕ, gives a concept of an admissible dissection for the polygon P. For a given P, consider the dissections $d \in D(P)$ such that $T_{\phi,d}(P)$ has only root and leaf vertexes. The edges corresponding to d define an admissible cut of $T_{\phi,d'}(P)$ for any $d' \in D(P)$ such that $d \subseteq d'$.

Definition 18. The dissection $c \in D(P)$ is an admissible dissection of P in the rule ϕ if $T_{\phi,c}(P)$ has only leaf or root vertexes.

This is rule specific. Let ψ be another rule generating a different algebra of trees. An admissible dissection of a polygon P in ϕ need to be an admissible dissection of ψ. For instance, the dissection $d = \{\alpha, \beta\}$ of the polygon

$P =$ 

is an admissible cut in ϕ_4 but not in ϕ_2.

Theorem 4. *Let ϕ be a rule generating the sub Hopf algebra $\mathfrak{T}_\phi \subset \mathfrak{T}^\bullet(R)$, and c is an admissible dissection of P in ϕ. Let $\{P_c^1, \dots, P_c^i\}$ be the labels of roots of $T_{\phi,c}(P)$ the decorations of the root vertexes, and $\{P_c^{i+1}, \dots, P_c^n\}$ the decorations of the leaf vertexes. Then*

$$\Delta(\Lambda_\phi(P)) = \sum_{c \text{ admis}} \amalg_{j=1}^{i} \Lambda_\phi(P_c^j) \otimes \amalg_{k=i+1}^{n} \Lambda_\phi(P_c^k) \; .$$

Proof. Fix an admissible dissection $c \in D(P)$ in $\mathfrak{T}_\phi$. Any dissection $d \in D(P)$ such that $c \subset d$ can be written as $d = c \cup_{j=1}^n d_i$. I have

$$\Delta \circ \Lambda_\phi(P) = \Delta \circ \Lambda(\sum_{d \in D(P)} T_{\phi,d}(P)) = (\Lambda \otimes \Lambda) \circ \Delta(\sum_{d \in D(P)} T_{\phi,d}(P)) \; ,$$

where the first equation comes from the definition of Λ_ϕ and the second equation from Theorem 1.

Write

$$\Delta T_{\phi,d}(P) = \sum_{c \text{ admis.}} R_d(c) \otimes L_d(c) \; ,$$

For $R_d(c) = \prod_{j=1}^{i} T_{\phi,d_j}(P_c^j)$ and $L_d(c) = \prod_{k=i+1}^{n} T_{\phi,d_k}(P_c^k)$ the root and leaf forests of $T_{\phi,d}(P)$ under the admissible cut c. Therefore

$$(\Lambda \otimes \Lambda)(\sum_{d \in D(P)} \sum_{c \subset d \text{ admis.}} \prod_{j=1}^{i} T_{\phi,d_j}(P_c^j) \otimes \prod_{k=i+1}^{n} T_{\phi,d_k}(P_c^k)) =$$

$$\sum_{c \text{ admis. diss.}} Ш_{j=1}^{i} \sum_{d_j \in D(P_c^j)} \Lambda(T_{\phi,d_j}(P_c^j)) \otimes Ш_{k=i+1}^{n} \sum_{d_k \in D(P_c^k)} \Lambda(T_{\phi,d}(P_c^k)) =$$

$$\sum_{c \text{ admis. diss.}} Ш_{j=1}^{i} \Lambda(\phi(P_c^j)) \otimes Ш_{k=i+1}^{n} \Lambda(\phi(P_c^k)) .$$

□

1.5. **Properties of algebras of trees that define differentials .** In this section, I show that if ϕ is a rule that defines the differential ∂ on $\mathcal{P}_\bullet^{(*)}(R)$, and $\phi(V(R))$ generates the algebra $\mathfrak{T}_\phi$, then Λ_ϕ is a Hopf algebra, even if $\mathfrak{T}_\phi$ is not.

First I show a relationship between the sums of linear orders of trees that differ on the orientation of the edges connecting certain vertexes, and linear orders of certain sub-trees.

Definition 19. Let I be a subset of the edges of a tree T. Let T^I be the tree obtained from T by reversing the orientation of the edges in I.

Lemma 4. *Let $T \in \mathfrak{T}^\bullet(R)$ be a decorated multi-rooted tree. Let I be a subset of n edges of the tree T. Let $F = t_1 \cdot t_2 \cdot \ldots t_{n+1}$ be the forest of multi-rooted trees created by removing the edges in I in T. Then*

$$\sum_{k \in 2^I} \Lambda(T^k) = \Lambda(F) := Ш_{i=1}^{n+1} \Lambda(t_i) .$$

Notice that I do not require I to be an admissible cut of T.

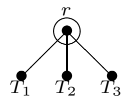
Proof. Let v_{j1} and v_{j2} be the endpoints of the edge $I_j \in I$ of T such that each edge I_j flows from v_{j1} to v_{j2}. Then in T^I $v_{j2} \prec v_{j1}$. Similarly, for any $k \subset I$, $v_{j2} \prec v_{j1}$ in T^k if $I_j \in k$ and from $v_{j1} \prec v_{j2}$ in T^k otherwise.

If t_k and t_l are two trees in the forest F, the vertices of t_k are incomparable to the vertices of t_l in the linear order defined by F. Furthermore, by construction, each tree t_k does not have both v_{j1} and v_{j2} as vertices. Therefore, I can group $\Lambda(F)$ into sums of those terms where v_{j1} is to the left of v_{j2} and sums of those where the opposite is true. The relative positions of v_{j2} and v_{j1} correspond to the two orientations of the edge I_j. Since there are 2 choices for each pair, this divides the terms of $\Lambda(F)$ into 2^n sums. Thus $\Lambda(F)$ has been grouped into the sums in the statement of the lemma. □

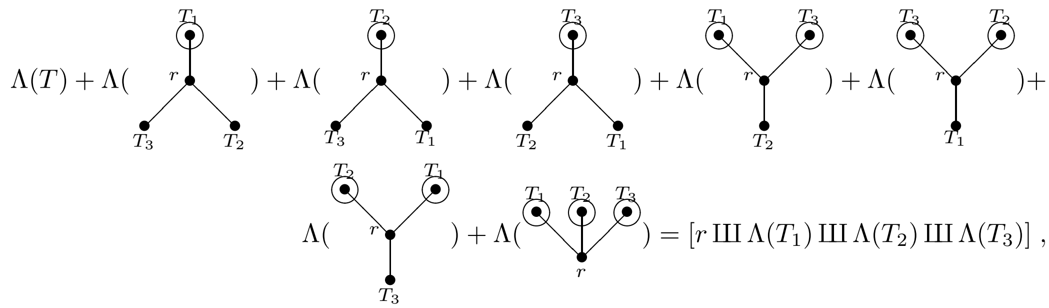
Example 12. (1) If $I = \{e\}$ is a single edge of a tree T, then let P_e be the pruned part of the cut I and let R_e be the root part. Then

$$\Lambda(T) + \Lambda(T^e) = \Lambda(R_e)\, Ш\, \Lambda(P_e)$$

(2) Let $T = B_r^s(\{T_i, v_{i2}\})$ be multi-rooted tree formed by connecting the trees in the forest $F = \prod_{i=1}^{n} T_i$ to a new root with label s at the vertex v_{i2}. For $i = 3$,

$T =$ 

then



which I can write

$$\begin{aligned}\Lambda(T) + [\Lambda(T_1)|r|\Lambda(T_2) \text{Ш} \Lambda(T_3)] + [\Lambda(T_2)|r|\Lambda(T_1) \text{Ш} \Lambda(T_3)] + [\Lambda(T_3)|r|\Lambda(T_1) \text{Ш} \Lambda(T_2)] \\ +[\Lambda(T_2) \text{Ш} \Lambda(T_3)|r|\Lambda(T_1)] + [\Lambda(T_1) \text{Ш} \Lambda(T_3)|r|\Lambda(T_2)] + [\Lambda(T_1) \text{Ш} \Lambda(T_2)|r|\Lambda(T_3)] \\ +[\Lambda(T_1) \text{Ш} \Lambda(T_2) \text{Ш} \Lambda(T_3)|r] = [r \text{Ш} \Lambda(T_1) \text{Ш} \Lambda(T_2) \text{Ш} \Lambda(T_3)] \ .\end{aligned}$$

Suppose ϕ is a rule that defines a differential ∂_ϕ and generates the sub Hopf algebra $\mathfrak{T}_\phi \in \mathfrak{T}^\bullet(R)$. Let ψ be another rule such that $\partial_\psi = \partial_\phi$. The the sum in Lemma 4 simplifies to relate the Hopf algebra $\Lambda(\mathfrak{T}_\phi)$ to the algebra $\Lambda(\mathfrak{T}_\psi)$.

Theorem 5. *For ϕ and ψ as above, let s be the set of arrows on which the two rules do not agree. For $d \in D(P)$, let $v(d) = \{P_d^1, \dots, P_d^{|d|+1}\}$ be the labels of $T_{\phi,d}(P)$. Then*

$$\Lambda_\phi(P) - \Lambda_\psi(P)) = \sum_{d \subseteq s, d \in D(P)} (-1)^{|d|} sign_\phi(d) \text{Ш}_{j=1}^{|d|+1} \Lambda_\phi(P_d^j)) \ . \tag{5}$$

for all $P \in V(R)$.

Proof. For any $d \in D(P)$, let $I(d) = d \cap s$. The trees $T_{\phi,d}(P)$ and $T_{\psi,d}(P)$ are related by changing signs and switching the orientation of edges corresponding to arrows in s. That is,

$$(-1)^{|I(d)|} T_{\phi,d}(P) = T_{\psi,d}^{I(d)}(P) \ .$$

Notice that

$$T_{\phi,d}(P) = T_{\psi,d}(P) \Leftrightarrow I(d) = \emptyset \ . \tag{6}$$

The left hand side of (5) can be written

$$\sum_{d' \in D(P)} \text{sign}_\phi(d') \left((-1)^{|I(d')|} \Lambda(T_{\phi,d'}^{I(d')}) - \Lambda(T_{\phi,d'}^{\emptyset}) \right) \ .$$

By equation (6), one can ignore the dissections d' that don't intersect the set s,

$$\sum_{d' \in D(P), I(d') \neq \emptyset} \text{sign}_\phi(d') \left((-1)^{|I(d')|} \Lambda(T_{\phi,d'}^{I(d')}) - \Lambda(T_{\phi,d'}^{\emptyset}) \right) \ . \tag{7}$$

The right hand side of (5) can be written

$$\sum_{d \subseteq s, d \in D(P)} (-\text{sign}_\phi(d)) \text{Ш}_{j=1}^{|d|+1} \sum_{d_j \in D(P_d^j)} \text{sign}_\phi(d_j) \Lambda(T_{\phi,d_j} P_d^j) \ ,$$

where the forests of the form $\{T_{\phi,d_j} P_d^j\}$ corresponds to cutting every tree dual to a dissection $d' \in D(P)$, containing d, $T_{\phi,d'}(P)$ at the edges in d. Here, the edges in $d' \backslash d$ are grouped as dissections of the subpolygons associated to d,

$$d_j \in D(P_d^j), \quad \cup d_j = d' \setminus d \ .$$

By Lemma 4, one can rewrite the right hand side as

$$\sum_{d\subseteq s, d\in D(P)} \sum_{d\subseteq d'\in D(P)} (-1)^{|d|}(-\mathrm{sign}_O(d')) \left(\Lambda(T^{\emptyset}_{\phi,d'}(P)) + \sum_{\delta\subseteq d} \Lambda(T^{\delta}_{\phi,d'}(P))\right) .$$

Each $d' \in D(P)$, contains (possibly empty) sets of the form $\delta \subseteq d'$, $\delta \subset s$. Therefore, the term $\Lambda(T^{\delta}_{\phi,d'}(P))$ appears once, with sign $(-1)^{|d|}$, for each dissection d contained in $I(d')$, containing δ, i.e. in the set $\{d|\delta \subseteq d \subseteq d' \cap s\}$. By summing over the size of such d, I reorganize the right hand side of (5)

$$\sum_{d'\in D(P), I(d')\neq\emptyset} \mathrm{sign}_\phi(d') \sum_{\delta\subseteq d, |d|=0}^{|I(d')|} (-1)^{|d|} \binom{|I(d')\setminus\delta|}{|d\setminus\delta|} \Lambda(T^{\delta}_{\phi,d'}(P)) =$$
$$\sum_{d'\in D(P), I(d')\neq\emptyset} \mathrm{sign}_\phi(d') \left((-1)^{|I(d')|}\Lambda(T^{I(d')}_{\phi,d'}(P)) - \Lambda(T^{\emptyset}_{\phi,d'}(P))\right) ,$$

to match (7). The first term on the right hand side corresponds to the case where $\delta = I(d')$ is the full set of backwards arrow in d'. The second term corresponds to the case where $\delta = \emptyset$. All other terms cancel.

□

Equation (5) defines an isomorphism between $\Lambda(\mathfrak{T}_\psi)$ and $\Lambda(\mathfrak{T}_\phi)$. To see that the relation is invertible, consider a polygon P, such that $d \cap s = \emptyset$ for all $d \in D(P)$. Then $\Lambda_\phi(P) = \Lambda_\psi(P)$. For a polygon P, define the integer $n = \max\{|d \cap s||d \in D(P)\}$. The inverse of the map defined in (5) is defined on polygons P inductively on n.

This shows that the algebras $\Lambda(\mathfrak{T}_\phi)$ and $\Lambda(\mathfrak{T}_\psi)$ have the same underlying vector space. The algebra generators $\Lambda_\phi(P)$ and $\Lambda_\psi(P)$ define different bases of this vector space. Specifically, Λ_ψ is also a Hopf algebra, and I show the following corollary.

Corollary 1. *Let ϕ and ψ be two rules that define ∂_ϕ, and let the algebra generated by $\phi(V(R))$, $\mathfrak{T}_\phi$ be a Hopf algebra. Then $\Lambda(\mathfrak{T}_\psi)$ is a sub Hopf algebra of $B_{D_\phi}(\mathcal{P}(R))$.*

Proof. By Theorem 5, write

$$\Delta\Lambda_\psi(P) = \Delta\Lambda_\phi(P) + \Delta \sum_{d\subseteq s, d\in D(P)} \prod_{\alpha\in d} (-\mathrm{sign}_\phi(\alpha)) Ш_{j=1}^{|d|+1} \Lambda_\phi(P^j_d) . \tag{8}$$

Since $\Lambda(\mathfrak{T}_\phi)$ is a Hopf algebra, the right hand side is a term in $\Lambda(\mathfrak{T}_\phi) \otimes \Lambda(\mathfrak{T}_\phi)$, call it

$$\sum_i Ш_{j_i}\Lambda_\phi(Q_{j_i})) \otimes Ш_{k_i}\Lambda_\phi(R_{k_i}));.$$

But generators of $\Lambda(\mathfrak{T}_\phi)$ can be written as a sum of shuffles of generators of $\Lambda(\mathfrak{T}_\psi)$,

$$\Lambda_\phi(P) = \sum_i Ш_{j_i}\Lambda_\psi(P_{j_i}) .$$

Therefore, the left hand side of (8) is in $\Lambda(\mathfrak{T}_\psi) \otimes \Lambda(\mathfrak{T}_\psi)$ and $\Lambda(\mathfrak{T}_\psi)$ is a Hopf algebra. □

Let s be the set of arrows where the rules ϕ and ψ above differ. If, for all polygons $P \in V(R)$, containment of a dissection $d \in D(P)$, $d \subset s$, implies that the $T_{\phi,s}(P)$ and $T_{\psi,s}(P)$ are linear, then the result of Theorem 5 simplifies greatly.

Corollary 2. *If in addition to the conditions for ϕ and ψ above, $T_{\phi,d}(P)$ is linear for all $P \in V(R)$, and all $d \in D(P)$ such that $d \subset s$, then*

$$\Lambda_\psi(P) - \Lambda_\phi(P) = \sum_{\alpha\in s} -sign_\phi(\alpha)\Lambda_\psi(P^1_\alpha) Ш \Lambda_\phi(P^2_\alpha) ,$$

Proof. Fix an $\alpha \in s$. Consider all $d \subseteq s$ such that α is dual to the (only) edge attached to the root in $T_{\phi,d}(P)$. Then the dissection $\{d \setminus \alpha\} \in D(P^2_\alpha)\}$. Let

$$\rho_\alpha = \{d \in D(P) | d \subset s, \alpha \in d, P^1_\alpha \text{ root label of } T_{\phi,d}(P)\}$$

be the set of all such d. As before, write $v(d) = \{P^1_d, \ldots, P^{|d|+1}_d\}$ and let $P^1_\alpha = P^1_d$ be the label of the root vertex of $T_{\phi,d}(P)$. From the previous theorem, the left hand side gives

$$\sum_{d \subseteq s} (-1)^{|d|} \mathrm{sign}_\phi(d) Ш_{j=1}^{|d|+1} \Lambda_\phi(P^j_d) = \sum_{\alpha \in s} \sum_{d \in \rho_\alpha} (-1)^{|d|} \mathrm{sign}_\phi(d) \Lambda_\phi(P^1_\alpha) Ш_{j=2}^{|d|+1} \Lambda_\phi(P^j_d) \ .$$

If $d = \alpha$, then $P^2_d = P^2_\alpha$. Break the sum in the right hand side of the previous equation down as

$$\sum_{\alpha \in s} -\mathrm{sign}_\phi(\alpha) \sum_{d \in \rho_\alpha} \Lambda_\phi(P^1_\alpha) Ш \left(\Lambda_\phi(P^2_\alpha) + \prod_{\beta \in d \setminus \alpha} (-\mathrm{sign}_\phi(\beta)) \sum_{d \in \rho_\alpha, d \neq \alpha} Ш_{j=2}^{|d|+1} \Lambda_\phi(P^j_d) \right) \ .$$

But for dissections $d \neq \alpha$,

$$\sum_{d \in \rho_\alpha} -\mathrm{sign}_\phi(d \setminus \alpha) Ш_{j=2}^{|d|+1} \Lambda_\phi(P^j_d) = \Lambda_\psi(P^2_\alpha) - \Lambda_\phi(P^2_\alpha) \ .$$

This gives

$$\Lambda_\psi(P) - \Lambda_\phi(P) = \sum_{\alpha \in s} -\mathrm{sign}_\phi(\alpha) \Lambda_\phi(P^1_\alpha) Ш \left(\Lambda_\phi(P^2_\alpha) + \Lambda_\psi(P^2_\alpha) - \Lambda_\phi(P^2_\alpha) \right) \ .$$

□

Corollary 3. *If $T_{\phi,s}(P)$ is linear, then*

$$\Lambda_\psi(P) - \Lambda_\phi(P) = \sum_{\alpha \in s} -sign_\phi(\alpha) \Lambda_\phi(P^2_\alpha) Ш \Lambda_\psi(P^1_\alpha) \ .$$

Proof. This result comes from running the same argument as above, with P^2_α the label of the unique leaf vertex of $T_{\phi,s}(P)$. □

2. Permutations of a polygon

In this section, I examine the actions of σ and τ on the Hopf algebra Λ_{ϕ_2}. Recall that σ and τ are linear automorphisms on $V(R)$ such that for $P = 12 \ldots n$, $\tau(P) = (n-1) \ldots 21n$ reverses the orientation of P and $\sigma(P) = 2 \ldots n1$ rotates the labels of the edges one position. Restricted to a subvectorspace $V_n(R)$ generated by polygons of weight n, $\sigma|_{V_n(R)}$ and $\tau|_{V_n(R)}$ generate the dihedral group D_{2n+2}. I can extend σ and τ to automorphisms of $\Lambda(\mathfrak{T}_2)$ by defining $\sigma(\Lambda_{\phi_2}(P)) = \Lambda_{\phi_2}(\sigma P))$ and $\tau(\Lambda_{\phi_2}(P)) = \Lambda_{\phi_2}(\tau P)$. After defining relations between $\Lambda_{\phi_2}(\sigma P)$, $\Lambda_{\phi_2}(\tau P)$ and $\Lambda_{\phi_2}(P)$, one can use the coalgebra homomorphism

$$\Phi : \Lambda(\mathfrak{T}_2) \to \mathcal{I}(R)$$

to establish relationships between iterated integrals with the appropriate dihedral action on the arguments.

2.1. **Order 2 generator of the dihedral group.** First I calculate $\Lambda_{\phi_2}(P) \pm \Lambda_{\phi_2}(\tau P)$. Since τ fixes the label of the root side of the polygon P, is it useful to examine an rule that differs from ϕ_2 on arrows ending on the root side that also defines the differential ∂. Specifically, I also need the map ϕ_{re} discussed in Example 10.

The rule ϕ_{re} differs from the rule ϕ_2 only when defining the order and signs of adjacent polygons connected by edges associated to arrows in $re(P)$. In section 1.4, I show that ϕ_{re} and ϕ_2 define the same differential operator. Therefore, $\Lambda(\mathfrak{T}_2)$ and $\Lambda(\mathfrak{T}_{re})$ are both Hopf algebras which can be related by Corollary 2.

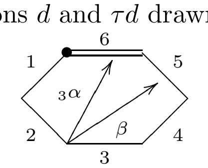
Lemma 5. *If P is an R-deco polygon of weight n,*

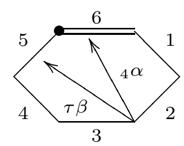
$$\Lambda_{\phi_2}(P) - \Lambda_{\phi_{re}}(P) = \sum_{i=2}^{n} \Lambda_{\phi_2}(P^{\bullet}_{i\alpha})) Ш \Lambda_{\phi_{re}}(P^{\sqcup}_{i\alpha}) \ .$$

Proof. For $i < j$, ${}_j\alpha \in D(P^{\sqcup}_{i\alpha})$ and ${}_i\alpha \in D(P^{\bullet\!=}_{j\alpha})$. If $d = \{{}_i\alpha, {}_j\alpha\} \in D(P)$, $T_{\phi_{re},d}(P)$ is linear. Therefore $T_{\phi_{re},re(P)}(P)$ is linear. The arrows ${}_j\alpha$ are forwards, so $\mathrm{sign}_{\phi_2}({}_j\alpha) = 1$. The result follows from Corollary 2. $\square$

Consider the action of τ on dissecting arrow and the associated subpolygons. Let α be a dissecting arrow of P. If $\alpha \neq re(P)$, that is $\alpha = {}_i\alpha_j$ (for $j \neq n+1$), the reflection map τ maps the polygon P to $\tau(P)$ and the arrow α to the arrow $\tau\alpha = {}_{n-i+2}\alpha_{n-j+1} \in D(\tau(P))$. For a root ending arrow, $\alpha = {}_i\alpha_{n+1} \in re(P)$ the arrow $\tau\alpha = {}_{n-i+2}\alpha_{n+1} \in D(\tau(P))$. For a forward (backward) arrow $\alpha \notin re(P)$, the arrow $\tau\alpha$ is backward (forward). All arrows in $re(P)$ are forward arrows. The following is an example for a 4-gon P. If $d \in D(P)$, write $\tau(d) \in D(\tau(P))$ the dissection of $\tau(P)$ written $\tau(d) = \{\tau\alpha_1, \ldots, \tau\alpha_k\}$, for $d = \{\alpha_1 \ldots \alpha_k\}$.

Example 13. Let $P = 123456$ be a 6-gon, and $d = \{{}_3\alpha, \beta\}$. Then $\tau d = \{{}_4\alpha, \tau\beta\}$. Below are diagrams of P and τP with the dissections d and τd drawn in.

$P =$ 

$\tau P =$ 

Here, the arrows ${}_3\alpha, {}_4\alpha \in re$. The subpolygons associated to ${}_3\alpha$ and ${}_4\alpha$ are

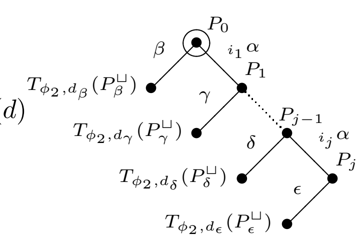
$$\tau(P^{\bullet\!=}_{3\alpha}) = 216 = (\tau P)^{\sqcup}_{\tau 3\alpha} \quad ; \quad (\tau P)^{\bullet\!=}_{\tau 3\alpha} = 5436 = \tau(P^{\sqcup}_{3\alpha})$$

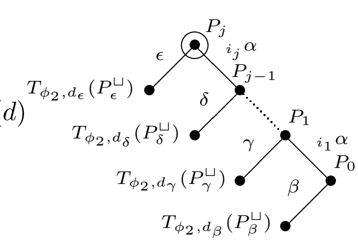
The subpolygons associated to β are

$$(\tau P)^{\bullet\!=}_{\tau\beta} = 5216 = \tau(P^{\bullet\!=}_{\beta}) \quad ; \quad P^{\sqcup}_{\beta} = 345 = (\tau P)^{\sqcup}_{\tau\beta} \; .$$

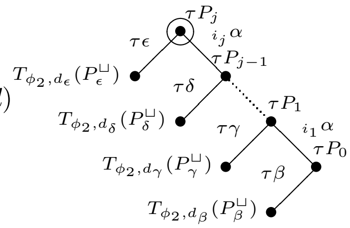
For a general dissection of an arbitrary polygon, $d \in D(P)$, such that $|d| = k$ and $d \cap re(P) = \{{}_{i_1}\alpha \ldots {}_{i_j}\alpha\}$, let $v(d) = \{P_0, \ldots P_k\}$ be the set of polygons labeling the vertexes of $T_{\phi_2,d}(P)$ and $T_{\phi_{re},d}(P)$, with the polygons P_{m-1} and P_m labeling the endpoints of the edge corresponding to ${}_{i_m}\alpha \in d \cap re(P)$. Then $v(\tau d) = \{\tau P_0, \ldots \tau P_j, P_{j+1}, \ldots, P_k\}$ is the set of polygons labeling tree $T_{\phi_2,\tau d}(\tau P)$. In $T_{\phi_2,d}(P)$, $P_{m-1} \prec P_m$. In $T_{\phi_2,\tau d}(\tau P)$, $\tau P_{m-1} \prec \tau P_m$, while $P_m \prec P_{m-1}$ in $T_{\phi_{re},d}(P)$. If $d \cap re(P) = \emptyset$ then P_0 is the label of the single root of all three trees.

For each $\beta \in d \setminus re(P)$ let $d_\beta = d \cap D(P^{\sqcup}_{\beta})$. In the following examples, assume β, γ, δ, and ϵ are all forwards arrows. If not, replace $P^{\sqcup}_{\beta}$ with $\tau(P^{\sqcup}_{\beta})$ and d_β with τd_β.

$T_{\phi_2,d}(P) = \mathrm{sign}_{\phi_2}(d)$ 

$T_{\phi_{re},d}(P) = (-1)^j \mathrm{sign}_{\phi_2}(d)$ 

$$T_{\phi_2,\tau d}(\tau P) = \mathrm{sign}_{\phi_2}(\tau d)$$



The trees $T_{\phi_2,\tau d}(\tau P)$ and $T_{\phi_{re},d}(P)$ represent the same partial order on different set of vertexes; the trees differ only by the labels of the vertexes. The vertices of the trees $T_{\phi_2,d}(P)$ and $T_{\phi_{re},d}(P)$ are both labeled by the same set of polygons, $v(d)$, though they represent different partial orders on that set. This is why the rule ϕ_{re} is a good intermediate map for comparing $\Lambda_{\phi_2}(\tau P)$ and $\Lambda_{\phi_2}(P)$. For any dissection $d \in D(P))$,

$$\mathrm{sign}_{\phi_{re}}(d) = (-1)^{|d\cap re(P)|}\mathrm{sign}_{\phi_2}(d) = (-1)^{\sum_{\alpha\in d,\alpha \mathrm{bw}}\chi(\alpha)}(-1)^{|d\cap re(P)|}$$

and

$$\mathrm{sign}_{\phi_2}(\tau d) = (-1)^{\sum_{\alpha\in d\setminus re(P),\alpha \mathrm{fw}}\chi(\alpha)} \ .$$

I proceed in by comparing the algebras $\Lambda(\mathfrak{T}_2)$ and $\Lambda(\mathfrak{T}_{re})$ by comparing the generators $\Lambda_{\phi_2}(\tau P)$ and $\Lambda_{\phi_{re}}(P)$.

Definition 20. Let $I_n \subset \mathcal{P}^{(1)}_\bullet(R)$ be the linear subspace generated by $\{P+(-1)^n\tau P|P$ polygon of weight $n\}$. This is a primitive co-ideal in $B_\partial(\mathcal{P}^{(\star)}_\bullet(R))$.

Notice that $I_1 = 0$ is the trivial co-ideal.

Theorem 6. *Let $P = r_0r_1\dots r_n$ be an R deco polygon of weight n. Let I_n be the primitive co-ideal defined above. Define a set of quotient maps $q_n : \Lambda(\mathfrak{T}_2) \to \Lambda(\mathfrak{T}_2)/(\sum_{k=1}^n I_k)$. Then, for $P \in V_n(R)$,*

$$\Lambda_{\phi_{re}}(P) + (-1)^n\Lambda_{\phi_2}(\tau P) \in \ker q_n \ .$$

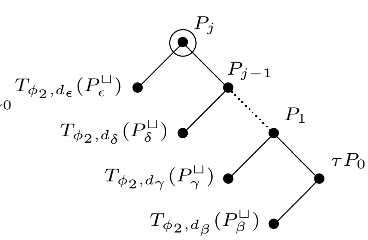
Proof. If P is a polygon of weight q, $\Lambda_{\phi_{re}}(P) - \Lambda_{\phi_2}(\tau P) = 0$. For $P = r_0r_1r_2 \in V_2(R)$,

$$\begin{aligned}\Lambda_{\phi_{re}}(P) + \Lambda_{\phi_2}(P) = P + [r_1r_2|r_0r_1] - [r_0r_2|r_1r_2] - [r_0r_2|r_1r_0] + \\ \tau P + [r_0r_2|r_1r_0] + [r_1r_2|r_0r_2] - [r_1r_2|r_0r_1] = P + \tau P \ .\end{aligned}$$

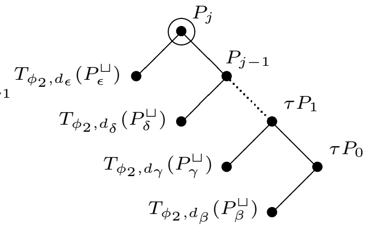
Suppose the theorem holds for all $k < n$.

Let P be an R-deco polygon of weight n. As before, consider a non-trivial dissection $d \in D(P)$, with $|d| = k \geq 1$ and $d \cap re(P) = \{{}_{i_1}\alpha \dots {}_{i_j}\alpha\}$. Let $v(d) = \{P_0, \dots P_k\}$ be the set of vertex labels of the trees $T_{\phi_{re},d}(P)$ and $T_{\phi_2,d}(P)$, with P_{m-1} adjacent to P_m in both trees, connected by ${}_{i_m}\alpha$, and $P_{m-1} \prec P_m$ in the former tree and $P_m \prec P_{m-1}$ in the latter. Suppose each $P_i \in v(d)$ is of weight n_i. I define a series of trees $\{T_{i,d}(P)\}$, $0 \leq i \leq j$ formed from $T_{i-1,d}(P)$ by replacing the polygon P_i with $(-1)^{n_i}P_i$. In this series, $T_{\phi_{re},d}(P) = T_{-1,d}(P)$. For example

$$T_{0,d}(P) = \mathrm{sign}_{\phi_{re}}(d)(-1)^{n_0}$$



$$T_{1,d}(P) = \mathrm{sign}_{\phi_{re}}(d)(-1)^{n_0+n_1}$$



.

For $0 \leq i \leq j$, $\Lambda(T_{i-1,d}(P) + T_{i,d}(P)) \in I_{n_i}$. By alternating sums,

$$\sum_{i=0}^{j}(-1)^i\Lambda(T_{i-1,d}(P) + T_{i,d}(P)) = T_{\phi_{re},d}(P) + (-1)^j T_{j,d}(P) \in \sum_{i=1}^{j} I_{n_i} \ .$$

Notice that

$$(-1)^j \mathrm{sign}(T_{j,d}) = (-1)^{\sum_{i=0}^{j} n_i}\mathrm{sign}_{\phi_2}(d) =$$
$$(-1)^{\sum_{i=0}^{j} n_i}(-1)^{\sum_{\alpha\in d, \alpha \mathrm{bw}}\chi(\alpha)} = (-1)^n \mathrm{sign}_{\phi_2}(\tau d) \ .$$

For all $d \in D(P)$, $d \neq \emptyset$,

$$(-1)^j T_{j,d}(P) = (-1)^n T_{\phi_2,\tau d}(\tau P) \ .$$

Applying the linearization map on these trees gives

$$\Lambda\left(\sum_{\substack{d\in D(P)\\ d\neq\emptyset}} T_{\phi_{re},d}(P) + (-1)^n T_{\phi_2,\tau d}(\tau P)\right) \in \sum_{k=1}^{n-1} I_k \ .$$

In other words,

$$\Lambda_{\phi_{re}}(P) - P + (-1)^n[\Lambda_{\phi_2}(\tau P) - \tau P] \in \ker q_{n-1} \ ,$$

and

$$\Lambda_{\phi_{re}}(P) + (-1)^n \Lambda_{\phi_2}(\tau P) \in \ker q_n \ .$$

□

Combining Theorem 6 with Lemma 5 gives the following.

Theorem 7. *If P is an R-deco polygon of weight n,*

$$q_n(\Lambda_{\phi_2}(P) + (-1)^n\Lambda_{\phi_2}(\tau P)) = q_n(\sum_{i=2}^{n}(-1)^{n-i}\Lambda_{\phi_{re}}(P^{\bullet}_{i\alpha}) ⧢ \Lambda_{\phi_{re}}(\tau(P^{\sqcup}_{i\alpha}))) \ .$$

That is, up to a primitive coideal, $\Lambda_{\phi_{re}}(P)$ can be related to $\Lambda_{\phi_{re}}(\tau P)$. This relation between decorated polygons of different orientation is reminiscent of a relation between iterated integrals on $R \subset \mathbb{C}^\times$. Recall that for iterated integrals, there is the relation

$$I(0; x_1, \ldots x_n; y)I(0; w_1, \ldots w_m; y) = I(0; (x_1, \ldots x_n) ⧢ (w_1, \ldots, w_m); y) \ .$$

Lemma 6. *Let $R \subset \mathbb{C}^\times$ be a set, and $r_i \in R$. Then*

(9)

$$I(0; r_1, \ldots, r_n; r_{n+1}) + (-1)^n I(0; r_n, \ldots r_0; r_{n+1}) = \sum_{i=2}^{n}(-1)^{n-i} I(0; r_1, \ldots, r_i; r_{n+1}) I(0; r_n, \ldots r_{i+1}; r_{n+1}) \ .$$

Proof. Compare the right hand side of (9) to

$$\sum_{i=2}^{n}(-1)^{n-i} I(0; (r_0, \ldots, r_i) ⧢ (r_{n-1}, \ldots r_{i+1}); r_n) \ .$$

For a fixed i each term in the shuffle product in equation (9) can be broken down into two groups, the terms where r_i comes before r_{i+1} and the terms where it comes after. The former cancel with a term in the shuffle

$$I(0;(r_1,\ldots,r_i,r_{i+1})\, ш\, (r_n,\ldots r_{i+2});r_{n+1})\ ,$$

and the latter in the shuffle

$$I(0;(r_1,\ldots,r_{i-1})\, ш\, (r_n,\ldots r_{i+1},r_i);r_{n+1})\ ,$$

both of which appear with signs opposite that of the fixed term. What remains are the terms (for $i=1$) $(-1)^n I(0;r_1,\ldots r_n;r_{n+1})$ and (for $i=n-1$) $I(0;r_n,\ldots r_1;r_{n+1})$, which match with the left hand side of (9). □

Remark 2. While on the level of polygons, there is a relation only up to a primitive coideal, the corresponding relation on iterated integrals is exact. Thus, for the map

$$\Phi:\Lambda(\mathfrak{T}_2)\to\mathcal{I}(R)\ ,$$

$\Phi(I_k)=0$ for all $k\geq 1$. Specifically, the coalgebra homomorphism Φ is not injective. The coideals $I_n\in\ker\Phi$.

I proceed with the hopes of finding a similar relation for the degree n generator, possibly up to a similar coideal.

2.2. **Order n generator of the dihedral group.** Now I consider the rotation map, σ on $\mathcal{P}(R)$ that sends the R deco polygon P to σP. If $P=12\ldots n$, $\sigma P=2\ldots n1$ is the polygon rotated clockwise, changing the root side. When restricted to $V_n(R)$, $\sigma|_{V_n(R)}$ is the order n generator of the dihedral group. In order to examine this rotation, I work with differentials that reflect the symmetry of the change, and relate the corresponding bar constructions to $\Lambda(\mathfrak{T}_2)$.

2.2.1. *Relating $\Lambda(\mathfrak{T}_2)$ to $\Lambda(\mathfrak{T}_4)$.* I want to calculate $\Lambda_{\phi_2}(P)-\Lambda_{\phi_2}(\sigma P))$. This is a difficult calculation, and it is easier to break down into intermediate steps. I use the results of the last section to relate the algebras $\Lambda_{\phi_2}(P)-\Lambda_{\phi_4}(P)$. I then study the action of σ on the algebra $\Lambda(\mathfrak{T}_4)$.

Definition 21. Let $b(P)=\{$ backwards arrows of $P\}$.

Recall that the rules ϕ_3 and ϕ_4 differ only by the order and sign of polygons at the endpoins of edges associated to backwards arrows.

Theorem 8. *Let $v(d)=\{P_d^1,\ldots,P_d^{|d|+1}\}$ be the set of polygons decorating the tree $T_{\phi_4,d}(P)$, for P weight n. One can write*

$$q_n\left(\Lambda_{\phi_2}(P)\right)=q_n\left(\Lambda_{\phi_4}(P)+\sum_{i=1}^{n-1}\sum_{d\subseteq b,|d|=i}(-1)^i\, ш_{j=1}^{i+1}\Lambda_{\phi_4}(P_d^j)\right)\ . \tag{10}$$

Proof. For all arrows $\alpha\in b(P)$, $\mathrm{sign}_{\phi_4}(\alpha)=+1$. By Theorem 5

$$\Lambda_{\phi_3}(P)=\Lambda_{\phi_4}(P)+\sum_{d\subseteq b}(-1)^{|d|}\, ш_{j=1}^{|d|+1}\Lambda_{\phi_4}(P_d^j)\ . \tag{11}$$

The trees $T_{\phi_2,d}(P)$ and $T_{\phi_3,d}(P)$ have different signs, but represent the same partial order on different sets. For $d\in D(P)$, with $|d\cap b(P)|=j$ and $|d|=k$ write

$$v(d)=\{P_d^1,\ldots P_d^{k+1}\}$$

the set of vertex labels of $T_{\phi_3,d}(P)$ and

$$w(d)=\{\tau P_d^1,\ldots\tau P_d^j,P_d^{j+1},\ldots P_d^{k+1}\}$$

the set of vertex labels of $T_{\phi_2,d}(P)$. Let $P_d^i\in V_i(R)$. I have enumerated the set of polygons such that P_d^i, for $1\leq i\leq j$, decorates the terminal vertex of the edge associated to a backwards arrow in $T_{\phi_3,d}(P)$.

The signs of d in the two rules are

$$\mathrm{sign}_{\phi_2}(d)=(-1)^{\sum_{i=1}^j n_i}\quad;\quad\mathrm{sign}_{\phi_3}(d)=(-1)^j\ .$$

As before, define a series of trees $\{T_{i,d}(P)\}$, $1 \leq i \leq j$ formed from $T_{i-1,d}(P)$ by replacing the polygon P_i with $(-1)^{n_i} P_i$. In this series, $T_{\phi_2,d}(P) = T_{0,d}(P)$, and $T_{j,d}(P) = (-1)^j T_{\phi_3,d}(P)$. Then

$$\sum_{i=1}^{j} (-1)^{j+1} \Lambda \left(T_{i-1,d}(P) + T_{i,d}(P) \right) = \Lambda \left(T_{\phi_2,d}(P) - T_{\phi_3,d}(P) \right) \in \sum_{k=1}^{i} I_{n_k} \ .$$

Varying the dissection d gives

$$\Lambda_{\phi_2,d}(P) + \Lambda_{\phi_3,d}(P) \in \sum_{k=1}^{n} I_k \ .$$

Plugging this into equation (11) gives

$$q_n \left(\Lambda_{\phi_2}(P)) \right) = q_n \left(\Lambda_{\phi_4}(P)) + \sum_{d \subseteq b, |d|=i} (-1)^{|d|} Ш_{j=1}^{|d|+1} \Lambda_{\phi_4}(P_d^j)) \right) \ .$$

□

Example 14. Let $P_2 = abc$, $P_3 = abcd$ be R-deco polygons of weight 2 and 3. The following are the explicit calculations for the 3-gon, 4-gon.

$$q_2(\Lambda_{\phi_2}(P_2)) - \Lambda_{\phi_4}(P_2))) = q_2(-\Lambda_{\phi_4}(ac) Ш \Lambda_{\phi_4}(ba))$$

This is actually an exact relation:

$$\Lambda_{\phi_2}(P_2)) - \Lambda_{\phi_4}(P_2)) = -\Lambda_{\phi_4}(ac) Ш \Lambda_{\phi_4}(ba)$$

For P_3,

$$\begin{aligned} q_3(\Lambda_{\phi_2}(P_3) - \Lambda_{\phi_4}(P_3))) = q_3(-\Lambda_{\phi_4}(acd) Ш \Lambda_{\phi_4}(ba) - \\ \Lambda_{\phi_4}(bca) Ш \Lambda_{\phi_4}(ad) - \Lambda_{\phi_4}(abd) Ш \Lambda_{\phi_4}(cb) + \Lambda_{\phi_4}(ad) Ш \Lambda_{\phi_4}(ba) Ш \Lambda_{\phi_4}(ca) + \\ \Lambda_{\phi_4}(ad) Ш \Lambda_{\phi_4}(ba) Ш \Lambda_{\phi_4}(cb) \end{aligned}$$

This is not an exact relation:

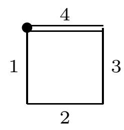
$$\begin{aligned} \Lambda_{\phi_2}(P_3) - \Lambda_{\phi_4}(P_3) = -\Lambda_{\phi_4}(acd) Ш \Lambda_{\phi_4}(ba) - \Lambda_{\phi_4}(bca) Ш \Lambda_{\phi_4}(ad) - \\ \Lambda_{\phi_4}(abd) Ш \Lambda_{\phi_4}(cb) + \Lambda_{\phi_4}(ad) Ш \Lambda_{\phi_4}(ba) Ш \Lambda_{\phi_4}(ca) + \\ \Lambda_{\phi_4}(ad) Ш \Lambda_{\phi_4}(ba) Ш \Lambda_{\phi_4}(cb) + [ad|bca] + [ad|cba] \end{aligned}$$

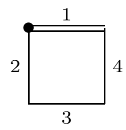
The algebra $\Lambda(\mathfrak{T}_4)$ is contained in $H^0(B_{\partial_3}(\mathcal{P}_\bullet^{(\star)}(R)))$, Theorem 3. The previous theorem also shows that $q_n(\Lambda_{\phi_4}(P)) \in H^0(B_{\partial_2}(\mathcal{P}_\bullet^{(\star)}(R)))$.

2.2.2. *Introducing a new symmetry.* Instead of directly trying to compare $\Lambda_{\phi_2}(P)$ and $\Lambda_{\phi_2}(\sigma P)$, I solve the easier problem of comparing $\Lambda_{\phi_4}(P)$ and $\Lambda_{\phi_4}(\sigma P)$.

Definition 22. If P is the R-deco polygon $12 \dots n$, with sides labeled 1 to n, let (σP) be the R-deco polygon $2 \dots n1$ with labels rotated one place mathematically negative orientation.

Example 15. For the weight 3 polygon $P = 1234$,



$= P$ and 

$= \sigma P$

Example 16. For a weight 1 polygon, $P = r_1 r_2$, $\sigma P = r_2 r_2$,

$$\Lambda_{\phi_4}(P) - \Lambda_{\phi_4}(\sigma P) = \Lambda_{\phi_2}(P) - \Lambda_{\phi_2}(\sigma P) =$$
$$\mathbb{L}\mathrm{i}(\frac{r_1}{r_2}) - \mathbb{L}\mathrm{i}(\frac{r_1}{r_2}) = \ln(r_1) - \ln(r_2)\ .$$

Direct calculation shows that for $P = r_1 r_2 r_3$,

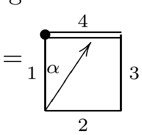
$$\Lambda_{\phi_4}(P) - \Lambda_{\phi_4}(\sigma P) = P - \sigma P + [r_1 r_2 ⧢ r_2 r_3] - [r_2 r_1 ⧢ r_3 r_2] + [(r_2 r_3 - r_3 r_2)|r_2 r_3] - [r_2 r_1|(r_2 r_1 - r_1 r_2)]\ .$$

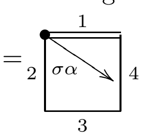
$$\Lambda_{\phi_2}(P) - \Lambda_{\phi_2}(\sigma P) =$$
$$P - \sigma P + [r_1 r_2 ⧢ r_2 r_3] - [r_1 r_3 ⧢ r_2 r_1] + [(r_2 r_3 - r_3 r_2)|r_2 r_3] - [r_2 r_1|(r_2 r_1 - r_1 r_2)]\ .$$

Subsequent direct calculations get increasingly complex.

To proceed, I examine the action of σ on the dissecting arrows of an R-deco polygon P. The rotation map σ acts on dissecting arrows, rotating the starting vertex and ending edge one position backwards, as defined by the orientation of the polygon. Therefore, $\sigma({}_i\alpha_j) = {}_{i-1}\alpha_{j-1}$ if i or $j \neq 1$, $\sigma_1\alpha_j = {}_{n+1}\alpha_{j-1}$ and $\sigma_i\alpha_1 = {}_{i-1}\alpha_{n+1}$.

Example 17. For the 4-gons P and σP, the dissecting arrows α and $\sigma\alpha$ are as follows:

$P =$ 

$\sigma P =$ 

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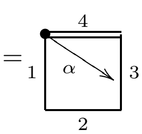
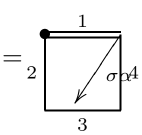
For a general $d \in D(P)$, write $d = \{\beta_1, \ldots \beta_k\}$ and $\sigma d = \{\sigma\beta_1, \ldots, \sigma\beta_k\}$. I want to study the structure of $\Lambda_{\phi_4}(P)$ versus $\Lambda_{\phi_4}(\sigma P)$. I start with dissections of P with one arrow. There are two cases to consider.

(1) The dissecting arrow α starts at the root vertex. The first vertex is in both P^r_α and in P^l_α. The associated subpolygon P are related to the subpolygons polygons of σP by

$$\sigma(P^r_\alpha) = (\sigma P)^r_{\sigma\alpha} \quad ; \quad \sigma(P^l_\alpha) = (\sigma P)^l_{\sigma\alpha}$$

$P =$ 

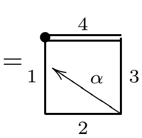
$\rightarrow P^l_\alpha = 34;\ P^r_\alpha = 123$ $\quad \sigma P =$ 

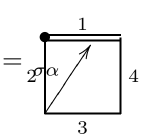
$\rightarrow (\sigma P)^l_{\sigma\alpha} = 43;\ (\sigma P)^r_{\sigma\alpha} = 231$.

(2) The dissecting arrow α does not start at the first vertex: There are three sub-cases.

(a) The dissecting arrow α ends on the first edge in P (labeled 1). The first vertex is in P^r_α. The dissected polygons of P and σP are

$$\sigma(P^r_\alpha) = (\sigma P)^r_{\sigma\alpha)} \quad ; \quad P^l_\alpha = (\sigma P)^l_{\sigma\alpha}$$

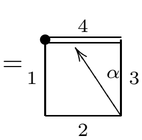
$P =$ 

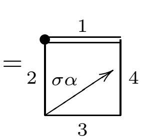
$\rightarrow P^l_\alpha = 21;\ P^r_\alpha = 134$ $\quad \sigma P =$ 

$\rightarrow (\sigma P)^l_{\sigma\alpha} = 21;\ (\sigma P)^r_{\sigma\alpha} = 341$.

(b) The dissecting arrow α ends on the root edge in P (labeled n). The first vertex is in P^l_α. The dissected polygons of P and σP are

$$P^r_\alpha = (\sigma P)^r_{\sigma\alpha} \quad ; \quad \sigma(P^l_\alpha) = \sigma P^l_{\sigma\alpha}$$

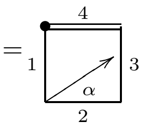
$P =$ 

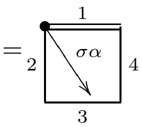
$\rightarrow P^l_\alpha = 124;\ P^r_\alpha = 34$ $\quad \sigma P =$ 

$\rightarrow \sigma P^l_{\sigma\alpha} = 241;\ (\sigma P)^r_{\sigma\alpha} = 34$.

(c) The dissecting arrow α ends on neither the first edge or root edge in P. The root vertex is in $P_\alpha^{l(r)}$ if α is forward (backward). The dissected polygons of P and σP are

$$\sigma(P_\alpha^{l(r)}) = (\sigma P)_{\sigma\alpha}^{l(r)} \quad ; \quad P_\alpha^{r(l)} = (\sigma P)_{\sigma\alpha}^{r(l)}$$

$P =$ 

$\rightarrow P_\alpha^l = 134;\ P_\alpha^r = 23 \quad \sigma P =$ 

$\rightarrow (\sigma P)_{\sigma\alpha}^l = 341;\ (\sigma P)_{\sigma\alpha}^r = 23$.

This exhaustively categorizes all possible dissecting arrows. I summarize the results as follows.

Lemma 7. *Let P be an R-deco polygon. The subpolygons of P associated to a single dissecting arrow can be classified in the following way:*

$$(\sigma P)_{\sigma\alpha}^r = \begin{cases} \sigma(P_\alpha^r) & \text{if } P_\alpha^r \text{ contains the root vertex of } P \\ P_\alpha^r & \text{otherwise} \end{cases}$$

The same is true if r is replaced with l. For arrows for the form ${}_1\alpha_j$, starting at the first vertex, both subpolygons P_α^r and P_α^l contain the first vertex.

I wish to calculate the action of the operator σ on the algebra $\Lambda(\mathfrak{T}_b)$. To proceed, I compare the algebra $\Lambda(\mathfrak{T}_4)$ to two new algebras $\Lambda(\mathfrak{T}_{fv})$ and $\Lambda(\mathfrak{T}_{\sigma fv})$, defined by new rules ϕ_{fv} and $\phi_{\sigma fv}$ that exploit the symmetries defined in Lemma 7.

Definition 23. Let $fv(P)$ be set of arrows that start at the $f$irst $v$ertex of an polygon P. If $P \in V_n(R)$, write $fv(P) = \{\alpha_2, \ldots, \alpha_{n+1}\}$ where α_i ends at the i^{th} side. Define $\sigma fv(\sigma P) = \{\sigma\alpha_2, \ldots, \sigma\alpha_{n+1}\}$ to be the set of arrows that start at the $n+1^{th}$ vertex of σP.

For $d \in D(P)$, with $|d| = k$, let $v(d) = \{P_d^0, \ldots P_d^k\}$ be the polygons decorating the tree $T_{\phi_{fv},d}(P)$ and $T_{\phi_4,d}(P)$. If $d \cap fv(P) = \{\alpha_{i_1}, \ldots, \alpha_{i_j}\}$, let P_d^{i-1} and P_d^i be the vertices decorating the edge associated to α_{i_j} in these trees. In $T_{\phi_4,d}(P)$, $P_d^{i-1} \prec P_d^i$, and $P_d^i \prec P_d^{i-1}$ in $T_{\phi_{fv},d}(P)$. If $d \cap fv(P) = \emptyset$, P_0 is the polygon associated to region of P containing the first vertex. Let

$$v(\sigma d) = \{\sigma P_d^0, \ldots, \sigma P_d^j, P_d^{j+1}, \ldots, P_d^k\}$$

be the set of polygons decorating the tree $T_{\phi_{\sigma fv},\sigma d}(\sigma P)$. The trees $T_{\phi_{fv},d}(P)$ and $T_{\phi_{\sigma fv},\sigma d}(\sigma P)$ define the same partial order on the two sets $v(d)$ and $v(\sigma d)$.

Remark 3. Let $\alpha \in fv(P)$. Then

$$\Delta_\alpha(P) = \Lambda_{\phi_{fv}}(P_\alpha^r) \otimes \Lambda_{\phi_{fv}}(P_\alpha^l)$$

and

$$\Delta_{\sigma\alpha}(\sigma P) = \Lambda_{\phi_{\sigma fv}}((\sigma P)_{\sigma\alpha}^r) \otimes \Lambda_{\phi_{fv}}((\sigma P)_{\sigma\alpha}^l) \ .$$

Otherwise, for $\beta \notin fv(P)$ suppose P_β^l contains the first vertex. Then

$$\Delta_\beta(P) = \Lambda_{\phi_4}(P_\beta^r) \otimes \Lambda_{\phi_{fv}}(P_\beta^l)$$

and

$$\Delta_{\sigma\beta}(\sigma P) = \Lambda_{\phi_4}((\sigma P)_{\sigma\beta}^r) \otimes \Lambda_{\phi_{fv}}((\sigma P)_\beta^l) \ .$$

Instead of calculating $\Lambda_{\phi_4}(P) - \Lambda_{\phi_4}(\sigma P)$, I calculate the expression

$$\big(\Lambda_{\phi_4}(P)) - \Lambda_{\phi_{fv}}(P))\big) - \big(\Lambda_{\phi_4}(\sigma P)) - \Lambda_{\phi_{\sigma fv}}(\sigma P))\big) + \big(\Lambda_{\phi_{fv}}(P)) - \Lambda_{\phi_{\sigma fv}}(\sigma P)\big) \ . \tag{12}$$

This is done in steps. The first two terms of (12) come from Corollary 2, and the hard part of this calculation is done in the last term.

Lemma 8. *Let P be an R-deco polygon of weight n.*

$$\big(\Lambda_{\phi_4}(P) - \Lambda_{\phi_{fv}}(P)\big) - \big(\Lambda_{\phi_4}(\sigma P) - \Lambda_{\phi_{\sigma f v}}(\sigma P)\big) =$$
$$\sum_{i=2}^{n-1} \Lambda_{\phi_4}(P^l_{\alpha_i}) \,Ш\, \Lambda_{\phi_{fv}}(P^r_{\alpha_i}) - \sum_{i=2}^{n-1} \Lambda_{\phi_4}((\sigma P)^l_{\sigma\alpha_i}) \,Ш\, \Lambda_{\phi_{fv}}((\sigma P)^r_{\bar{\alpha}_i}) \,.$$

Proof. For $\alpha_i, \alpha_j \in fv(P)$, with $i < j$, α_i dissects the subpolygon $P^r_{\alpha_j}$, and α_j dissects the subpolygon $P^l_{\alpha_i}$. Similarly, for $\sigma\alpha_i, \sigma\alpha_j \in \sigma fv(P)$, with $i < j$, $\sigma\alpha_i$ dissects the subpolygon $(\sigma P)^l_{\alpha_j}$, and $\sigma\alpha_j$ dissects the subpolygon $(\sigma P)^r_{\alpha_i}$. Therefore the trees $T_{\phi_{fv}, fv(P)}(P)$ and $T_{\phi_{\sigma fv}, \sigma fv(P)}(\sigma P)$ are linear, and the result follows form Corollary 2 and 3. $\square$

Example 18. For $P = 1234$,

$$\Lambda_{\phi_4}(P) - \Lambda_{\phi_{fv}}(P) =$$
$$\Lambda_{\phi_4}(234) \,Ш\, \Lambda_{\phi_{fv}}(12) + \Lambda_{\phi_4}(34) \,Ш\, \Lambda_{\phi_{fv}}(123) =$$
$$\Lambda_{\phi_4}(12) \,Ш\, \Lambda_{\phi_4}(234) + \Lambda_{\phi_4}(123) \,Ш\, \Lambda_{\phi_4}(34) - \Lambda_{\phi_4}(12) \,Ш\, \Lambda_{\phi_4}(23) \,Ш\, \Lambda_{\phi_4}(34) \,,$$

and for $\sigma P = 2341$

$$\Lambda_{\phi_4}(\sigma P) - \Lambda_{\phi_{fv}}(\sigma P) =$$
$$\Lambda_{\phi_4}(342) \,Ш\, \Lambda_{\phi_{fv}}(21) + \Lambda_{\phi_4}(231) \,Ш\, \Lambda_{\phi_{fv}}(34) =$$
$$\Lambda_{\phi_4}(21) \,Ш\, \Lambda_{\phi_4}(342) + \Lambda_{\phi_4}(231) \,Ш\, \Lambda_{\phi_4}(43) - \Lambda_{\phi_4}(43) \,Ш\, \Lambda_{\phi_4}(32) \,Ш\, \Lambda_{\phi_4}(21) \,,$$

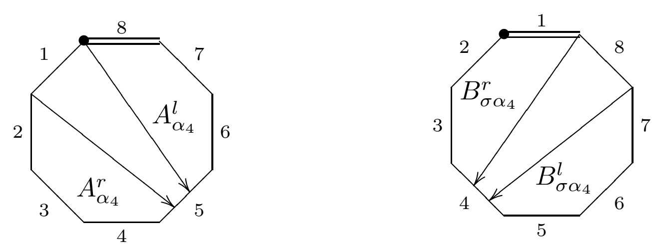
The third term in (12) is harder to prove.

Let $P = 1 \dots n+1$ be polygon of weight n and let $A = 2 \dots n+1$ and $B = 2 \dots n1$ be two polygons of weight $n-1$. The polygon A can be drawn as a subpolygon of P, with

$$A^l_{\alpha_i} = P^l_{\alpha_{i+1}}, \text{ and } A^r_{\alpha_i} = P^r_{{}_2\alpha_{i+1}} \,.$$

Similarly, B can be drawn as a subpolygon of σP with

$$B^l_{\sigma\alpha_i} = P^l_{{}_{n+1}\alpha_i} = (\sigma P)^l_{\sigma({}_{n+1}\alpha_i)}, \text{ and } B^r_{\sigma\alpha_i} = \sigma(P^r_{\alpha_i}) = (\sigma P)^r_{\sigma\alpha_i} \,.$$



For a fixed i, any dissection $d \in D(A)$ such that $\alpha_i \in d$, write $d = d_l \cup d_r \cup \alpha_i$, with $d_l \in D(A^l_{\alpha_i})$ and $d_r \in D(A^r_{\alpha_i})$. The set of polygons labeling the vertices of the tree $T_{\phi_{fv}, d}(A)$ is

$$v(d) = \{A^1_{d_r}, \dots A^j_{d_r}, A^1_{d_l}, \dots A^k_{d_l}\}$$

with $A^1_{d_l}$ and $A^1_{d_r}$ decorating the endpoints of the dual edge to α_i. There is a corresponding dissection, $d' \in D(P)$ such that the vertexes of $T_{\phi_{fv}, d'}(P)$ are

$$v(d') = \{A^1_{d_r}, \dots A^j_{d_r}, A^1_{d_l}, \dots A^k_{d_l}, 1i\} \,.$$

This tree can be written

$$T_{\phi_{fv}, d'}(P) = B^{1i}_r((T_{\phi_4, d_r}(A^r_{\alpha_i}), A^1_{d_r}), (T_{\phi_{fv}, d_l}(A^l_{\alpha_i}), A^1_{d_l}))$$

Similarly, for a fixed i, any dissection $\sigma\delta \in D(B)$ such that $\sigma\alpha_i \in \sigma\delta$, write $\sigma\delta = \sigma\delta_l \cup \sigma\delta_r \cup \sigma\alpha_i$, with $\sigma\delta_l \in D(B^l_{\sigma\alpha_i})$ and $\sigma\delta_r \in D(B^r_{\sigma\alpha_i})$. The set of vertexes labeling the tree $T_{\phi_{\sigma fv}, \sigma\delta}(B)$ is

$$v(\sigma\delta) = \{B^1_{\sigma\delta_r}, \dots B^j_{\sigma\delta_r}, B^1_{\sigma\delta_l}, \dots B^k_{\sigma\delta_l}\}$$

such that $B^1_{\sigma\delta_l}$ and $B^1_{\sigma\delta_r}$ decorate the endpoints of the dual edge to $\sigma\alpha_i$. There is a corresponding dissection, $\sigma\delta' \in D(P)$ such that the vertexes of $T_{\phi_{\sigma f v},d'}(P)$ are $\sigma\delta' \in D(\sigma P)$ such that

$$v(\sigma\delta') = \{B^1_{\sigma\delta_r}, \dots B^j_{\sigma\delta_r}, B^1_{\sigma\delta_l}, \dots B^k_{\sigma\delta_l}, (n+1)i\} \ .$$

This tree can be written

$$T_{\phi_{\sigma f v},\sigma\delta'}(P) = B_l^{(n+1)i}((T_{\phi_4,\sigma\delta_l}(B^l_{\alpha_i}), B^1_{\sigma\delta_l}), (T_{\phi_{fv},d_r}(B^r_{\alpha_i}), B^1_{\sigma\delta_r})) \ .$$

In the subsequent analysis, I will want fix i, and consider

$$\Lambda\left(\sum_{\alpha_i \in d \in D(A)} T_{\phi_{fv},d'}(P)\right) \ .$$

I define the root adjoining and leaf adjoining operators on trees to the algebras defined by rules.

Definition 24. Consider the polygons P_1, P_2, the rules ϕ_1, and ϕ_2, and dissections $d_i \in D(P_i)$. For each tree $T_{\phi_i,d_i}(P_i)$, choose a vertex $P^*_{d_i} \in v(d_i)$. Then write

$$\sum_{d_1 \in D(P_1)} \sum_{d_1 \in D(P_1)} \Lambda\left(B^s_r((T_{\phi_1,d_1}(P_1), P^*_{d_1}), (T_{\phi_2,d_2}(P_2), P^*_{d_2}))\right)$$
$$=: \Lambda B^s_r((\Lambda_{\phi_1}(P_1), \{P^*_{d_1}\})), (\Lambda_{\phi_2}(P_2), \{P^j_{d_2}\})) \ ,$$

and

$$\sum_{d_1 \in D(P_1)} \sum_{d_1 \in D(P_1)} \Lambda\left(B^s_l((T_{\phi_1,d_1}(P_1), P^*_{d_1}), (T_{\phi_2,d_2}(P_2), P^*_{d_2}))\right)$$
$$=: \Lambda B^s_l((\Lambda_{\phi_1}(P_1), \{P^*_{d_1}\}), (\Lambda_{\phi_2}(P_2), \{P^*_{d_2}\})) \ ,$$

The linear operators $\Lambda B^v_l, \Lambda B^v_r : \prod_k \mathcal{P}^{(\star)}_\bullet(R) \to B_\partial(\mathcal{P})$, and define Hochschild 1 cocycles. Details of this are found in Appendix A.

I now compute $\Lambda_{\phi_{fv}}(P)) - \Lambda_{\phi_{\sigma f v}}(\sigma P))$. Recall from Corollaries 2 and 3, this is an expression in the algebra $\Lambda(\mathfrak{T}_{\phi_4})$.

Definition 25. Let $J_n \subset \mathcal{P}^{(1)}_\bullet(R)$ be the linear subspace generated by $\{P - \sigma P | P \text{ polygon of weight } n\}$. Viewed as a subspace of the coalgebra $B_\partial(\mathcal{P})$, it is a primitive co-ideal.

Lemma 9. *Let P, A, $A^1_{d_r}$, $A^1_{d_l}$, B, $B^1_{\sigma\delta_r}$, $B^1_{\sigma\delta_l}$ be as above. For P an R-deco polygon of weight n, define the quotient map*

$$r_n : \Lambda(\mathfrak{T}_{\phi_4}) \to \Lambda(\mathfrak{T}_{\phi_4})/(\sum_{k=2}^n J_k) \ ,$$

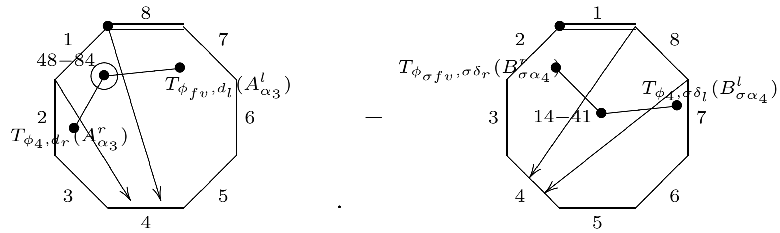
For $n \geq 2$,

$$r_n\left(\Lambda_{\phi_{fv}}(P) - \Lambda_{\phi_{\sigma f v}}(\sigma P))\right) = r_n\left(\sum_{i=2}^n \Lambda B^{i(n+1)-(n+1)i}_r((\Lambda_{\phi_4}(A^r_{\alpha_{i-1}}), \{A^1_{d_r}\}), (\Lambda_{\phi_{fv}}(A^l_{\alpha_{i-1}}), \{A^1_{d_l}\}))\right.$$
$$\left. - \sum_{i=2}^n \Lambda B^{i1-1i}_l((\Lambda_{\phi_4}(B^l_{\sigma\alpha_i}), \{B^1_{\sigma\delta_l}\}), (\Lambda_{\phi_{\sigma f v}}(B^r_{\sigma\alpha_i}), \{B^1_{\sigma\delta_r}\}))\right) \ . \tag{13}$$

Proof. The argument in the right hand side of (13) is a linearizations of the sum of trees

$$\sum_{i=2}^n B^{i(n+1)-(n+1)i}_r((\sum_{d_r \in D(A_r)} T_{\phi_4,d_r}(A^r_{\alpha_i}), A^1_{d_r}), (\sum_{d_l \in D(A_l)} T_{\phi_{fv},d_l}(A^l_{\alpha_i}), A^1_{d_l})) -$$
$$B^{1i-i1}_l((\sum_{d_l \in D(B_l)} T_{\phi_4,\sigma\delta_l}(B^l_{\alpha_i}), B^1_{\sigma\delta_l}), (\sum_{d_l \in D(A_l)} T_{\phi_{fv},d_r}(B^r_{\alpha_i}), B^1_{\sigma\delta_r})) \ .$$

For $i = 4$, this may look like



Since α_1 and α_n are trivial arrows, $A^r_{\alpha_1}$ and $B^l_{\alpha_n}$ are not defined. From this overlay of trees, notice that any admissible cut of a sum of trees of the form

$$B_r^{i(n+1)-(n+1)i}((T_{\phi_4,d_r}(A^r_{\alpha_i}), A^1_{d_r}), (T_{\phi_{fv},d_l}(A^l_{\alpha_i}), A^1_{d_l}))$$
$$-B_r^{i1-1i}((T_{\phi_4,\sigma\delta_r}(B^l_{\alpha_i}), B^1_{\sigma\delta_l}), (T_{\phi_{fv},\sigma\delta_l}(B^r_{\alpha_i}), B^1_{\sigma\delta_r}))$$

corresponds to an admissible cut on the left hand side (13).

Notice from Example 16 that this equation is trivial for $n = 2$ and holds for $n = 3$, since, for $P = 123$,

$$\Lambda_{\phi_{fv}}(P)) - \Lambda_{\phi_{\sigma fv}}(\sigma P)) = [23 - 32|23] - [21 - 12|21] + P - \sigma P\ .$$

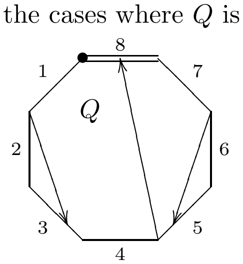
Suppose it holds for all m-gons for $m < n$. The edges of the trees involved in the right hand side of (13) correspond to non-empty dissections of P. If the theorem holds for all n, then J_n is a coideal of Λ_{ϕ_4}. This proof continues by comparing the coproducts of the right and left side of (13).

Let c be an admissible dissection of P in ϕ_{fv}. Since the tree $T_{\phi_{fv},fv(P)}(P)$ is linear, c contains at most one arrow in fv.

Fixed such a c, such that $c \cap fv(P) = \emptyset$. In this case, there is only one label of $T_{\phi_{fv},c}(P)$ that inherits its root vertex from P. Call this subpolygon of weight q, Q, and the corresponding subpolygon of σP, σQ. Write $Q = 1a_2 \dots a_q(n+1)$; Q and σQ have the same root side labels as P and σP. The trees $T_{\phi_{fv},d}(P)$ and $T_{\phi_{\sigma fv},\sigma d}(\sigma P)$ are identical after replacing Q with σQ. The term $\Lambda_{\phi_{fv}}(Q) - \Lambda_{\phi_{\sigma fv}}(\sigma Q)$ appears in either the right or left tensor component of $\Delta_c\left(\Lambda_{\phi_{fv}}(P)) - \Lambda_{\phi_{\sigma fv}}(\sigma P))\right)$. If $c = c' \cup \{{}_2\alpha_{a2}, {}_{a_q+1}\alpha_{n+1}\}$, for some dissection c', where the arrows $\{{}_2\alpha_{a2}, {}_{a_q+1}\alpha_{n+1}\}$ may be trivial, then Q is a root subpolygon. If $c = c' \cup \{{}_{a_2}\alpha_1, {}_{n+1}\alpha_{a_q}\}$, where the arrows $\{{}_{a_2}\alpha_1, {}_{n+1}\alpha_{a_q}\}$may be trivial, Q is a leaf subpolygon. By remark 3, I can write

$$(14)\qquad \Delta_c\left(\Lambda_{\phi_{fv}}(P)) - \Lambda_{\phi_{\sigma fv}}(\sigma P))\right) = \begin{cases} Ш_i\Lambda_{\phi_4}(R_i) \otimes \Lambda_{\phi_{fv}}(Q) - \Lambda_{\phi_{\sigma fv}}(\sigma Q)\,Ш_j\Lambda_{\phi_4}(L_j) & \text{or} \\ \Lambda_{\phi_{fv}}(Q) - \Lambda_{\phi_{\sigma fv}}(\sigma Q)\,Ш_i\Lambda_{\phi_4}(R_i) \otimes Ш_j\Lambda_{\phi_4}(L_j) & , \end{cases}$$

where R_i and L_j are labels of root or leaf polygons of $T_{\phi_{fv},c}(P)$. The following arguments are identical for the cases where Q is a root subpolygon or a leaf subpolygon. I only present the root case.

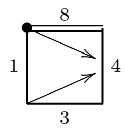


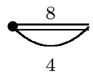
$Q = 1348$

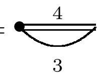
Let $A_Q = a_2 \dots a_q(n+1)$, and $B_Q = a_2 \dots a_q 1$ be subpolygons of A and B respectively. Define A^1_{Q,d_r}, A^1_{Q,d_l}, $B^1_{Q,\sigma\delta_r}$, $B^1_{Q,\sigma\delta_l}$ on A_Q and B_Q as above. By induction, $\Delta_c\left(\Lambda_{\phi_{fv}}(P)) - \Lambda_{\phi_{\sigma fv}}(\sigma P))\right)$ can be written (with $a_1 = 1$)

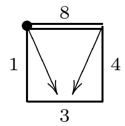
$$\sum_{i=2}^{q} \Lambda B_r^{a_i(n+1)-(n+1)a_i}\left((\Lambda_{\phi_4}(A^r_{Q\alpha_{i-1}}), \{A^1_{Q,d_r}\}), (\Lambda_{\phi_{fv}}(A^l_{Q\alpha_{i-1}}), \{A^1_{Q,d_l}\})\right) -$$

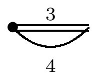
$$(15)\qquad \sum_{i=1}^{q-1} \Lambda B_l^{a_i1-1a_i}\left((\Lambda_{\phi_4}(B^l_{Q\sigma\alpha_i}), \{B^1_{Q,\sigma\delta_l}\}), (\Lambda_{\phi_{fv}}(B^r_{Q\sigma\alpha_i}), \{B^1_{Q,\sigma\delta_r}\})\right)\ .$$

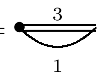


; $A^l_{Q_{\alpha_2}} =$ 

$A^r_{Q_{\alpha_2}} =$ 



; $B^l_{Q_{\alpha_2}} =$ 

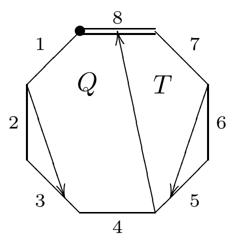
$B^r_{Q_{\alpha_2}} =$ 

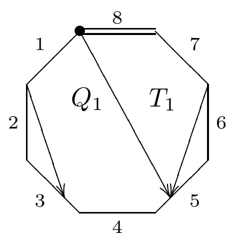
Plugging (15) into the first line of (14) gives a summand of the coproduct of the right hand side of (13).

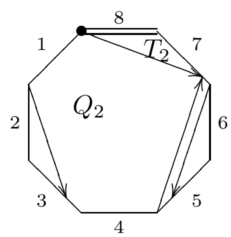
Let the polygon $T = [(a_q + 1)b_2 \ldots b_t(n + 1)]$ be the polygon associated to the region to the right of ${}_{a_q+1}\alpha_{n+1}$. Write $b_1 = a_q + 1$. Next consider a related set of admissible cuts the contain an arrow in $fv(P)$,

$$e_m = (c \setminus {}_{a_q+1}\alpha_{n+1}) \cup \{\alpha_{b_m}, {}_{a_q+1}\alpha_{b_i}\} = c' \cup \{{}_2\alpha_{a2}, \alpha_{b_m}, {}_{a_q+1}\alpha_{b_i}\} \ ,$$

with $i \leq t$, a decoration of the polygon T, as described above. The arrows $\{{}_2\alpha_{a2}, {}_{a_q}\alpha_{b_i}\}$ may be trivial.



; 

; 

$Q_1 = 1345 \quad T_1 = 578 = T$
$Q_2 = 1346 \quad T_2 = 78$

Define three polygons, S_m, T_m, Q_m, decorating $T_{\phi_{fv}, e_m}(P)$: $Q_m = 1a_2 \ldots a_q b_m$ is the polygon of weight q corresponds to the region in P to the right of the arrow α_{b_m} and to the left of ${}_{a_q+1}\alpha_{b_m}$, the polygon $T_m = b_m \ldots b_t(n + 1)$ corresponds to the region to the left of α_{b_m}, the polygon $S_m = b_1 \ldots b_m$ corresponds to the region in P to the right of the arrow ${}_{a_q+1}\alpha_{b_m}$. If $m = 1$, S_1 is not defined, and $T_1 = T$ as defined above. The polygons S_m and T_m, if they exists, are adjacent to Q_m.

The contribution to the coproduct of e_m is

$$\begin{aligned}
&-\Delta_{e_m}(\Lambda_{\phi_{fv}}(P) - \Lambda_{\phi_{\sigma fv}}(\sigma P)) = \\
&Ш_k \Lambda_{\phi_4}(R_k)\, Ш \left(\Lambda_{\phi_{fv}}(Q_m) - \Lambda_{\phi_{\sigma fv}}(\sigma Q_m)\right) \otimes [\Lambda_{\phi_{fv}}(T_m)\, Ш\, \Lambda_{\phi_4}(S_m)]\, Ш_j \Lambda_{\phi_4}(L_j) \\
&+ Ш_k \Lambda_{\phi_4}(R_k)\, Ш\, \Lambda_{\phi_{\sigma fv}}(\sigma Q_m) \otimes \left[\left(\Lambda_{\phi_{fv}}(T_m) - \Lambda_{\phi_{\sigma fv}}(\sigma T_m)\right) Ш\, \Lambda_{\phi_4}(S_m)\right] Ш_j \Lambda_{\phi_4}(L_j) \ .
\end{aligned} \tag{16}$$

The negative sign comes from the sign of $T_{\phi_{fv}, e_m}(P)$. The last line of (16) is relevant to calculations where Q is a leaf polygon, and is ignored for now.

Consider the terms in sum

$$(\Delta_c + \sum_{m=1}^{t} \Delta_{e_m}) \left(\Lambda_{\phi_{fv}}(P)) - \Lambda_{\phi_{\sigma fv}}(\sigma P))\right) \ .$$

From Corollary 2,

$$\Lambda_{\phi_4}(T) = \sum_{m=1}^{t} \Lambda_{\phi_{fv}}(T_m) \ .$$

Inserting this into the sum gives

$$\begin{aligned}
&Ш_k \Lambda_{\phi_4}(R_k)\, Ш \left(\Lambda_{\phi_{fv}}(Q) - \Lambda_{\phi_{\sigma fv}}(\sigma Q)\right) \otimes \\
&\left(\sum_{m=1}^{t} \Lambda_{\phi_{fv}}(T_m)\, Ш\, \Lambda_{\phi_4}(S_m)\right) Ш_j \Lambda_{\phi_4}(L_j) \\
-&\sum_{m=1}^{t} Ш_k \Lambda_{\phi_4}(R_k)\, Ш \left(\Lambda_{\phi_{fv}}(Q_m) - \Lambda_{\phi_{\sigma fv}}(\sigma Q_m)\right) \otimes \\
&[\Lambda_{\phi_{fv}}(T_m)\, Ш\, \Lambda_{\phi_4}(S_m)]\, Ш_j \Lambda_{\phi_4}(L_j) \ .
\end{aligned}$$

Collecting like terms gives

$$\sum_{m=1}^{t} Ш_k \Lambda_{\phi_4}(R_k)\, Ш \left(\Lambda_{\phi_{fv}}(Q-Q_m) - \Lambda_{\phi_{\sigma fv}}(\sigma Q - \sigma Q_m)\right) \otimes$$
$$[\Lambda_{\phi_{fv}}(T_m)\, Ш\, \Lambda_{\phi_4}(S_m)]\, Ш_j \Lambda_{\phi_4}(L_j)\ .$$

Define $A_{Q_m} = [a_2 \dots a_q b_j]$ the subpolygons of Q_j, with $A_{Q_0} := A_Q$. Define $A^1_{Q_m,d_r}$, $A^1_{Q_m,d_l}$, on A_{Q_m} as above. Using the induction step in (15), this becomes

$$\sum_{m=1}^{t}\left(\sum_{i=2}^{q} \Lambda B_r^{a_i(n+1)-(n+1)a_i}\left((\Lambda_{\phi_4}(A^r_{Q\alpha_{i-1}}),\{A^1_{Q_{d_r}}\}),(\Lambda_{\phi_{fv}}(A^l_{Q\alpha_{i-1}}),\{A^1_{Q_{d_l}}\})\right) Ш_k\Lambda_{\phi_4}(R_k)\right.$$
$$\left. - \Lambda B_r^{a_i b_m - b_m a_i}\left((\Lambda_{\phi_4}(A^r_{Q_m,\alpha_{i-1}}),\{A^1_{Q_m,d_r}\}),(\Lambda_{\phi_{fv}}(A^l_{Q_m,\alpha_{i-1}}),\{A^1_{Q_m,d_l}\})\right)\right) Ш_k\Lambda_{\phi_4}(R_k)$$
$$\otimes[\Lambda_{\phi_{fv}}(T_m)\, Ш\, \Lambda_{\phi_4}(S_m)]\, Ш_j \Lambda_{\phi_4}(L_j)\ . \tag{17}$$

The terms involving $B_l^{a_i 1 - a_i 1}$ cancel because

$$B_Q = B_{Q_j} = a_2 \dots a_q 1$$

for all j.

The terms in the first line of (17) are of the desired form. Using the relation

$$[ab]-[ba] = \ln a - \ln b + \ln c - \ln c = [ac]-[ca]+[cb]-[bc],$$

rewrite the terms in the second line as

$$-\sum_{m=1}^{t}\sum_{i=2}^{q} \Lambda B_r^{a_i(n+1)-(n+1)a_i}\left((\Lambda_{\phi_4}(A^r_{Q_m,\alpha_{i-1}}),\{A^1_{Q_m,d_r}\}),(\Lambda_{\phi_{fv}}(A^l_{Q_n,\alpha_{i-1}},\{A^1_{Q_m,d_l}\})\right) Ш_k\Lambda_{\phi_4}(R_k)$$
$$\otimes[\Lambda_{\phi_{fv}}(T_m)\, Ш\, \Lambda_{\phi_4}(S_m)]\, Ш_j \Lambda_{\phi_4}(L_j)$$
$$\sum_{m=1}^{t}\sum_{i=2}^{q} \Lambda B_r^{b_m(n+1)-(n+1)b_m}\left((\Lambda_{\phi_4}(A^r_{Q_m,\alpha_{i-1}}),\{A^1_{Q_m,d_r}\}),(\Lambda_{\phi_{fv}}(A^l_{Q_m,\alpha_{i-1}},\{A^1_{Q_m,d_l}\})\right) Ш_k\Lambda_{\phi_4}(R_k)$$
$$\otimes[\Lambda_{\phi_{fv}}(T_m)\, Ш\, \Lambda_{\phi_4}(S_m)]\, Ш_j \Lambda_{\phi_4}(L_j) \tag{18}$$

Taking the sum over all admissible cuts that don't intersect $fv(P)$, these terms cancel, as the arrow ${}_{q+1}\alpha_{n+1}$ varies in c.

Similar arguments show that admissible cuts c such that Q is a root label in $u(c)$ pair with the last line of (14) to give the appropriate expression involving the leaf adjoining operator. □

Appendix A. Hochschild cycles

Let A be a bialgebra. Let $C_n(A, A^{\otimes n}) = \hom(A, A^{\otimes n})$ be the n^{th} co-chain, with the coboundary b. Define

$$\Delta^i = (\underbrace{id\otimes\ldots\otimes id}_{i-1 \text{ times}}\otimes\Delta\otimes id\ldots\otimes id)$$

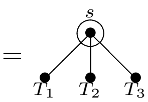
to be the operator from $A^{\otimes n} \to A^{\otimes n+1}$ that takes the coproduct on the i^{th} component. If $\varphi \in C_n$,

$$b\varphi = (id\otimes\varphi)\Delta + \sum_{i=1}^{n-1}(-1)^i\Delta^i\circ\varphi + (-1)^{n+1}(\varphi\otimes 1)\ .$$

Coassociativity of Δ ensures that $b\circ b = 0$. Similarly, for a non-cocommutative bialgebra, one can define a twisted boundary map

$$\tau b\varphi = (\varphi\otimes id)\Delta + \sum_{i=1}^{n-1}(-1)^i\Delta^i\circ\varphi + (-1)^{n+1}(1\otimes\varphi)\ .$$

For a set of single rooted trees $\{T_1, \ldots T_n\}$, there is an operator $B_r^s(T_1, \ldots, T_n)$ that combines the trees T_i into a new tree by defining a new root, with label s, and an edge from s to the root of each tree T_i. For example

$B_r^s(T_1, T_2, T_3) =$ 

.

If $A \subset \mathfrak{T}^\bullet(R)$ is the bialgebra of single rooted trees decorated by R-deco polygons, [1] show that $B_r^s(A)$ is a Hochschild 1-cocycle for the twisted boundary map τb. The key part of the proof lies in checking the relation

$$\Delta B_r^s = (B_r^s \otimes id)\Delta + 1 \otimes B_r^s \; . \tag{19}$$

One can similarly define an operator B_l^s on the subalgebra $\mathcal{A} \in \mathfrak{T}^\bullet(R)$ of multi rooted, single leafed trees. In this case, check that

$$\Delta B_l^s = (id \otimes B_l^s)\Delta + B_l^s \otimes 1.$$

This result comes from the equivalence of roots and leaves in the algebra $\mathfrak{T}^\bullet(R)$. Switching the orientation of all the edges (and thus the role of root and leaf) in a single rooted tree gives a multi rooted, singles leafed tree. This amounts to switching the order in which the terms appear in the coproduct on $\mathfrak{T}^\bullet(R)$, which is the action of the twisting operator.

In this paper, I have extended these operators to all of $\mathfrak{T}^\bullet(R)$, $B_r^s((T, v), (R, w))$ and $B_l^s((T, v), (R, w))$ connect a new root (leaf) with decoration s to the trees T and R by connecting s to the vertex v and w. Recall that the linearization map

$$\Lambda : \mathfrak{T}^\bullet(R) \to W(R)$$

is surjective.

Definition 26. Define ΛB_r^s and ΛB_l^s as operators on $W(R)$ such that

$$\Lambda B_r^s((\Lambda T, v), (\Lambda R, w)) = \Lambda(B_r^s((T, v), (R, w)))$$

is the sum of all linear orders of the forest $T \; R \; s$ such that s appears before v and w. Similarly,

$$\Lambda B_l^s((\Lambda T, v), (\Lambda R, w)) = \Lambda(B_l^s((T, v), (R, w)))$$

is the sum of all linear orders of the forest $T \cdot R \cdot s$ such that s appears after v and w.

In this appendix, I show that ΛB_r^s and ΛB_l^s are both Hochschild 1-cocycles on $W(R)$ under the coboundary maps τb and b respectively.

Definition 27. Let $A \subset \mathfrak{T}^\bullet(R)$ be the subalgebra of single rooted trees. Let $\mathcal{A} \subset \mathfrak{T}^\bullet(R)$ be the subalgebra of multi rooted, single leafed trees.

Define an operator $B_r^{s,\{v_i\}} : A \to A$ on the algebra of single rooted trees. It maps a forest of single rooted trees with k vertexes to a tree with $k+1$ vertexes. In the forest $\prod_{j \in J} T_j$, consider the sub forest indexed by $I \subset J$ comprised of trees T_i such that the root vertex of T_i is in the set $\{v_i\}$.

Definition 28. The operator

$$B_r^{s,\{v_i\}}(T_{j \in J}) = (\prod_{J \setminus I} T_j) B_r^s(T_{j \in I})$$

acts as the identity on trees that don't have *root* vertex labels in the predetermined set $\{v_i\}$. Define $B_r^{s,\{v_i\}}(1) = \bullet^s$ for all sets $\{v_i\}$. Similarly, extend

$$B_l^{s,\{v_i\}}(T_{j \in J}) = (\prod_{J \setminus I} T_j) B_l^s(T_{j \in I}),$$

to act as the identity on trees that don't have *leaf* vertex labels in the predetermined set $\{v_i\}$. Define $B_l^{s,\{v_i\}}(1) = \bullet^s$ for all sets $\{v_i\}$.

Lemma 10. (1) *The operator $B_r^{s,\{v_i\}}$ is a Hochschild 1-cocycle for the boundary map τb on A.*

(2) *The operator $\mathcal{B}_l^{s,\{v_i\}}$ is a Hochschild 1-cocycle for the boundary map b on $\mathcal{A}$.*

Proof. The identity map is a Hochschild 1-boundary, and B_r^s is a Hochschild 1-cocycle. Therefore

$$\Delta B_r^{s,\{v_i\}}(\prod_J T_j) = \Delta \prod_{J\setminus I} T_j \quad \Delta B_r^s(T_{j\in I}) =$$

$$\Delta \prod_{J\setminus I} T_j \quad ((B_r^s \otimes id)\Delta(T_{j\in I}) + 1 \otimes B_r^s(T_{j\in I})) = ((id \otimes B_r^{s,\{v_i\}})\Delta + B_r^{s,\{v_i\}} \otimes id)(\prod_J T_j) \ .$$

The proof for $\mathcal{B}_l^{s,\{v_i\}}$ is identical. □

Definition 29. Define maps $\Lambda B_r^{s,\{v_i\}}$ and $\Lambda B_l^{s,\{v_i\}}$ as operators on ΛA and $\Lambda\mathcal{A}$, as sub-algebras of $W(R)$, by

$$\Lambda B_r^{s,\{v_j\}}(\Lambda \prod_J T_j) = \Lambda B_r^{s,\{v_j\}}(\prod_J T_j) \ ,$$

and

$$\Lambda B_l^{s,\{v_j\}}(\Lambda \prod_J T_j) = \Lambda B_l^{s,\{v_j\}}(\prod_J T_j) \ .$$

Theorem 9. *The operators ΛB_r^s and ΛB_l^s are Hochschild 1-cocycles on the bialgebra $W(R)$ under the boundary maps τb and b respectively.*

Proof. The linearization map $\Lambda : \mathfrak{T}^\bullet(R) \to W(R)$ is surjective. Therefore, it is sufficient to study linearizations of $B_r^{s,\{v_i\}}$.

Since Λ is a bialgebra homomorphism, by Lemma 10, $\Lambda B_r^{s,\{v_i\}}$ and $\Lambda B_l^{s,\{v_i\}}$ are Hochschild 1-cocycles on ΛA and $\Lambda\mathcal{A}$ respectively.

Consider the case of $B_r^s(T, v)$, for a single tree $T \in \mathfrak{T}^\bullet(R)$. For any multi rooted tree, one can switch orientations of a subset of edges I of any multi rooted tree T to get a *single* rooted tree T^I with root vertex v. Let T^I be the tree with the *single* root vertex v formed from T by switching the edges in the sets I, and T' the forest of single rooted trees formed by removing the edges I in T. By Lemma 4,

$$\Lambda(T^I) = \Lambda(T') - \Lambda(T) - \sum_{k\subsetneq I} \Lambda(T^k) \ .$$

Now $B_r^s(T^I, v) = B_r^{s,v}(T^I)$.

Suppose I is a set of 1 element. Let $T' = T_1 \cdot T_2$, with v the root vertex of T_1. Then

$$\Lambda(B_r^s(T, v)) = \Lambda(B_r^{s,v}(T_1) \cdot T_2) - \Lambda(B_r^{s,v}(T^I)) \ .$$

Since the right hand side satisfies condition (19) on $\Lambda\mathfrak{T}^\bullet(R)$, so is $\Lambda B_r^s(\Lambda T, v)$. For a general I, let $T' = \prod_{i=1}^{|I|+1} T_i$, with v the root vertex of T_1. Then

$$\Lambda B_r^s(T, v) = \Lambda(B_r^{s,v}(T_1) \cdot \prod_{i=2}^{|I|+1} T_i) - \Lambda B_r^{s,v}(T^I) - \Lambda(\sum_{k\subsetneq I} B_r^s(T^k, v)) \ .$$

Since $|k| < I$, by induction the operator ΛB_r^s on $W(R)$ satisfies (19), so is $\Lambda B_r^s(\Lambda T, v)$. A similar proof shows that B_l^s on $W(R)$ can be written as a Hochschild 1-cocycle $\Lambda B_l^{s,*}$.

Now consider B_r^s acting on a forest $\prod_{j\in J}(T_j)$ with specified vertexes v_j. Let I_j be the set of edges of T_j that need to be reversed to get a single rooted tree, $T_j^{I_j}$ with root v_j. From the arguments above,

$$\Lambda B_r^s(\{T_j, v_j\}_J) = Ш_J \left(\Lambda(B_r^{s,v_j}(T_j')) - \Lambda B_r^{s,v}(T_j^{I_j}) - \Lambda(\sum_{k\subsetneq I_j} B_r^s(\{T_j^k, v_j\})) \right) \ .$$

Similar arguments show the theorem from $\Lambda B_l^{s,*}$ □

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